ANALYTICAL STUDY OF VIBRATION DATA REDUCTION METHODS

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ANALYTICAL STUDY OF VIBRATION DATA REDUCTION METHODS

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FOREWORD

This report is the culmination of a study made for the Vibration and Data Systems Section of the National Aeronautics and Space Administration, George C. Marshall Space Flight Center, Huntsville, Alabama, by the Technical Products Company of Los Angeles. It was prepared under contract NAS8-5093. The study was under the technical direction and administration of Jack A. Jones on the part of N.A.S.A. and Robert C. Moody on the part of the Technical Products Company. The authors were Julius S. Bendat, Loren D. Enochson and Allan G. Piersol. Many contributions were made by N.A.S.A. personnel and members of the Technical Products Company engineering staff. Particularly valuable contributions were made by Murl H. Newberry, Milton T. Herrin, Gerald J. Curet, Linn W. Hall, Richard E. Moore and Ronald E. Jewell of the Marshall Space Flight Center.

ABSTRACT

15527

This report discusses analog and digital data reduction methods for the analytical study of vibration data. Emphasis is placed on basic characteristics of vibration data to be analyzed. General techniques are developed appropriate to both periodic data and random data. Practical considerations are included for such matters as analog instrumentation parameters and digital sample sizes which limit accuracy of desired measurements. Material on the interpretation and evaluation of data is illustrated with numerous engineering examples to make the results as meaningful as possible.

The report is divided into three sections. Section 1, Data Reduction by Analog Methods, covers various analog techniques for data reduction which give measurements of amplitude probability density functions, correlation functions, and power spectral density functions. Statistical properties of random data, measurement accuracies, analog computer times, and calibration procedures are discussed in detail. Section 2, Interpretation and Evaluation of Data, discusses the analysis and application of vibration data to engineering problems. Tests for randomness, stationarity, and equivalence are outlined. Important relationships for amplitude probability density functions, correlation functions, and power spectral density functions, are then reviewed and applied to various engineering examples. Section 3, Theoretical Considerations and Digital Analysis of Data, discusses further theoretical matters involved in vibration data analysis. This is followed by mathematical formulae for carrying out a digital analysis of vibration data, including flow diagrams and estimates of required digital computer time to calculate the various functions discussed in earlier sections of the report.

CONTENTS

CONTENTS (Continued)

CONTENTS (Continued)

1. DATA REDUCTION BY ANALOG METHODS

In general, data reduction is the translation of raw information into simplified quantities which describe the information in a manner suitable for engineering analysis and interpretation. For the specific case of flight vehicle vibration data, the available raw information is usually one or more sample amplitude time history records of the vibration response at one or more points on the vehicle structure. Methods employed for reducing vibration data by analog methods are described here, along with appropriate discussions of some general considerations associated with vibration data analysis.

1.1 BASIC CHARACTERISTICS OF VIBRATION DATA

Before data reduction can be pursued in detail, it is necessary to identify certain basic characteristics of the data. In particular, it should be determined if the vibration data is representative of a random process as opposed to a periodic process or other deterministic processes. Perhaps the data represents a combination of both. The procedures for reducing, analyzing, and interpreting data representing a random vibration response are different from those for a periodic vibration response. The basic characteristics of periodic and random vibration data will now be discussed.

1.1.1 Periodic Vibration Data

A periodic function is a special type of deterministic (analytic) function whose amplitude time history repeats itself exactly after a time interval T_p called the period. In equation form, a necessary condition for a function $y(t)$ to be periodic is

$$y(t) = y(t + T_p) \tag{1.1}$$

This is in contrast to other types of deterministic functions and random processes where Eq.(1.1) becomes an inequality. That is,

$$y(t) \neq y(t + T_p) \tag{1.2}$$

A periodic function may be expressed by a Fourier series as shown in Eq. (1.3), where $f_1 = 1/T_p$ is the frequency in cycles per second (cps).

$$\begin{aligned} y(t) &= C_0 + C_1 \cos(2\pi f_1 t + \phi_1) + C_2 \cos(4\pi f_1 t + \phi_2) + C_3 \cos(6\pi f_1 t + \phi_3) + \cdots \\ &= C_0 + \sum_{n=1}^{\infty} C_n \cos(2\pi n f_1 t + \phi_n) \end{aligned} \tag{1.3}$$

In words, Eq.(1.3) says that a periodic function consists of a D.C. component, C_0, and an infinite number of sinusoidal components having amplitudes C_n and phases ϕ_n. The frequencies of the sinusoidal components are all even multiples of f_1, which is called the fundamental frequency. Many periodic functions consist of only a few or even a single component. For example, a sine wave has a Fourier series in which all values of C_n are zero except for $n = 1$. In other cases, the fundamental frequency is absent. For example, suppose a periodic function is formed by mixing three sine waves which have frequencies of 60, 75, and 100 cps. The lowest common multiple is 5 cps, so the period for the resulting periodic function is $T_p = 0.2$ seconds. Hence, in the Fourier series for the function, all values of C_n are zero except for $n = 12$, $n = 15$, and $n = 20$.

It should be noted that many deterministic functions are not periodic. For example, suppose we have the sum of three sine waves which have frequencies of 60, 75, and $300/\pi$. There is no common multiple since $300/\pi$

is an irrational number, so the resulting function is not periodic. However, in actual practice, such nonperiodic functions may be closely approximated by a periodic function and expanded into an approximate Fourier series.

1.1.2 Random Vibration Data

Unlike periodic functions, the amplitude time history for a random function never repeats itself exactly. Any given sample record represents a unique set of circumstances, and is merely a special example out of a large set of possible records that might have occurred. The collection of all possible records that might have occurred is called an ensemble which forms a random process. Thus, an amplitude time history record for a random vibration response may be thought of as a sample record from a random process.

Since a random process is probabilistic and not an explicit function of time, the prediction of exact amplitudes at some future time is not possible. Thus, a random process must be described in terms of statistical averages as opposed to exact analytic functions. It is for this reason that the techniques for reducing, analyzing, and interpreting random vibration data are different from those for periodic vibration data.

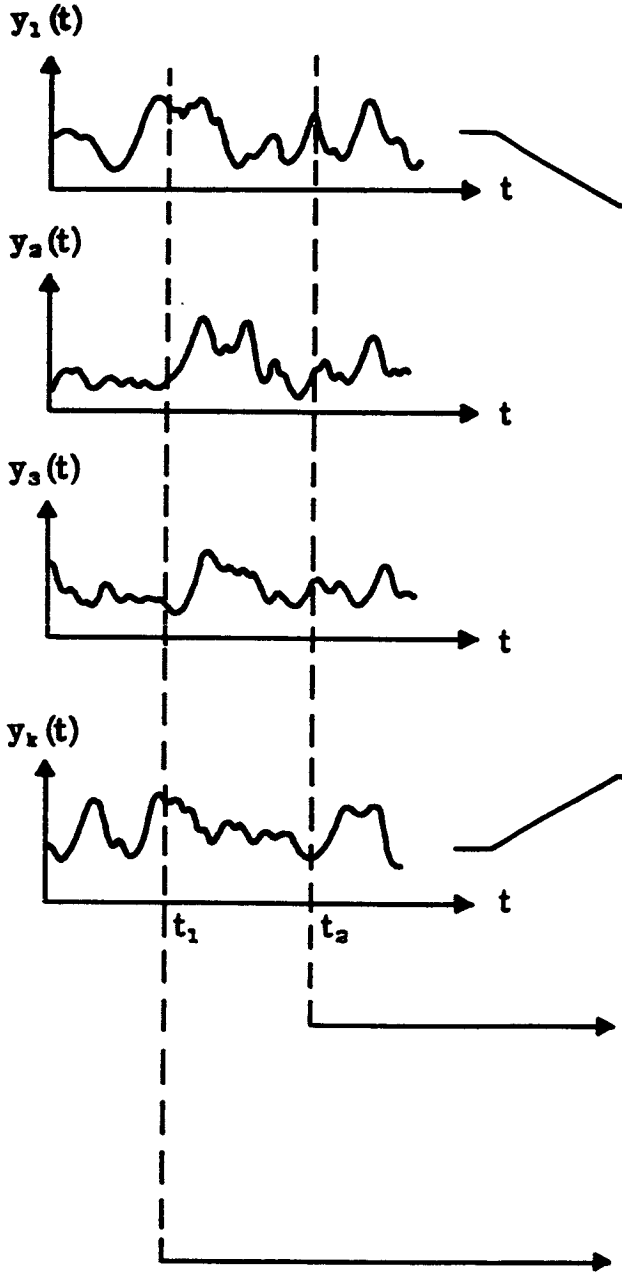
The properties of a random process may be computed by taking averages over the ensemble at any given time t. For example, the mean value (first moment) of a random process at some time t_1 is computed by taking the instantaneous amplitude of each record of the ensemble at time t_1, summing the amplitudes, and dividing by the number of records. This

computation is illustrated in Figure 1-1. The mean square value (second moment) at time t_1, and the correlation between amplitudes at two different times t_1 and t_2 are computed in a similar manner as illustrated in Figures 1-2 and 1-3.

For the general case where the mean value, mean square value, and correlation function vary with time, the random process is said to be nonstationary. For the special case where these three properties do not vary with time, the random process is said to be weakly stationary. If all higher moments, that is the third moment and up, are also time invariant, the random process is said to be strongly stationary.

If a random process is stationary, the properties of single records in the random process can be computed by taking time averages of the single records as opposed to ensemble averages for the collection of records. The computation of the mean value, mean square value, and correlation function by time averaging is illustrated in Figures 1-1 through 1-3. For the general case where these three time averaged properties vary from record to record, the random process is said to be nonergodic. For the special case where these three time averaged properties do not vary from record to record and thus are equal to the corresponding ensemble averaged properties, the random process is said to be weakly ergodic. If all higher moments are also independent of the record used, the random process is said to be strongly ergodic. Note that only stationary random processes can be ergodic.

If a vibration response is representative of a nonstationary random process, the properties of the vibration are changing with time and can be



$\langle\ \rangle$ Indicates ensemble average

— Indicates time average

$$\bar{y}_i = \lim_{T \to \infty} \frac{1}{T} \int_0^T y_i(t)\, dt$$

= time average for record $y_i(t)$

$$\bar{y}_k = \lim_{T \to \infty} \frac{1}{T} \int_0^T y_k(t)\, dt$$

= time average for record $y_k(t)$

$$\langle y(t_2) \rangle = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} y_i(t_2)$$

= ensemble average for time $t = t_2$

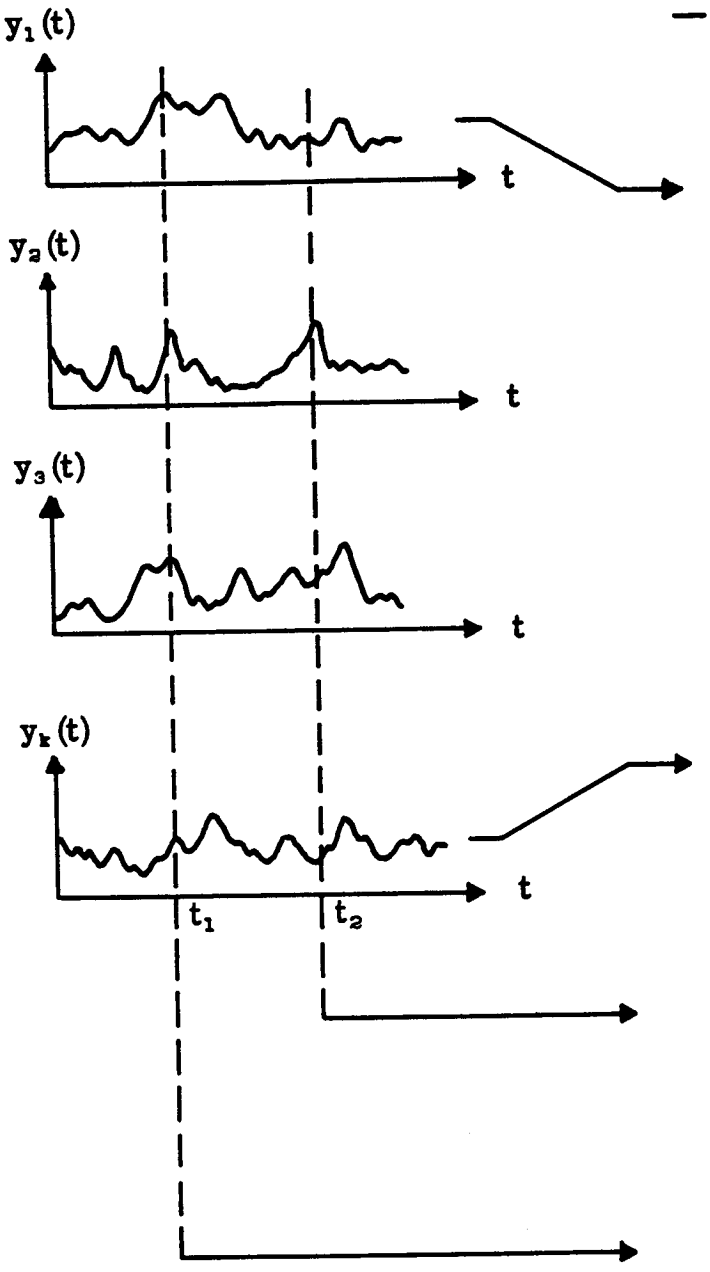
$$\langle y(t_1) \rangle = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} y_i(t_1)$$

= ensemble average for time $t = t_1$

If the random process is stationary, $\langle y(t_1) \rangle = \langle y(t_2) \rangle$

If the stationary random process is ergodic, $\langle y(t) \rangle = \bar{y}_i;\ i = 1, 2, 3, \ldots$

FIGURE 1-1. COMPUTATION OF MEAN VALUES



$\langle\ \rangle$ Indicates ensemble average

— Indicates time average

$$\overline{y_i^2} = \lim_{T \to \infty} \frac{1}{T} \int_0^T y_i^2(t)\, dt$$

= mean square time average for record $y_1(t)$

$$\overline{y_k^2} = \lim_{T \to \infty} \frac{1}{T} \int_0^T y_k^2(t)\, dt$$

= mean square time average for record $y_k(t)$

$$\left\langle y^2(t_2) \right\rangle = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} y_i^2(t_2)$$

= mean square ensemble average for time $t = t_2$

$$\left\langle y^2(t_1) \right\rangle = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} y_i^2(t_1)$$

= mean square ensemble average for time $t = t_1$

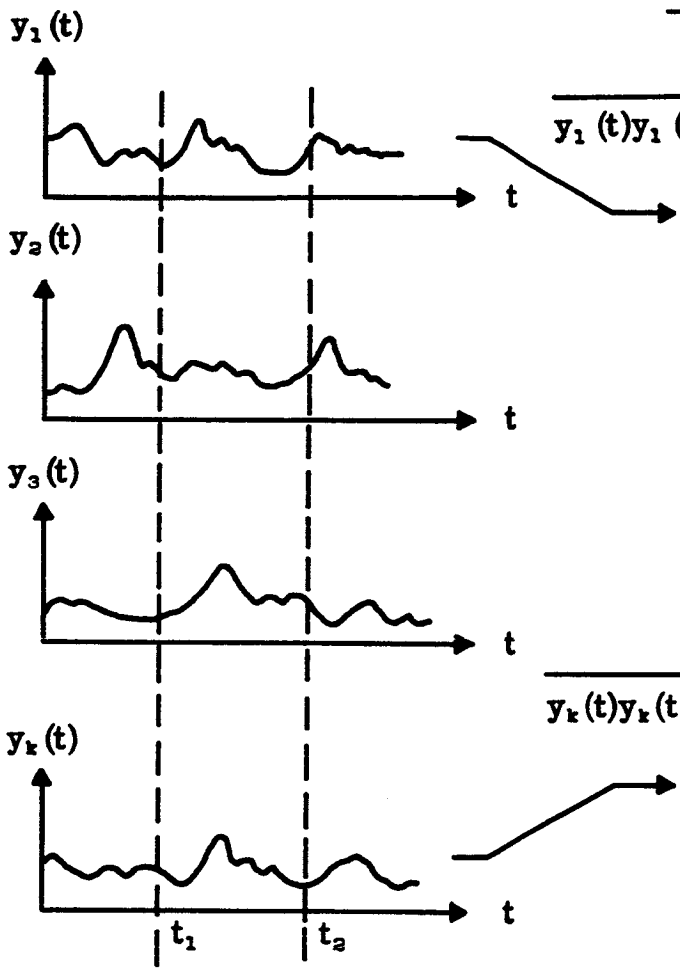
If the random process is stationary, $\left\langle y^2(t_1) \right\rangle = \left\langle y^2(t_2) \right\rangle$

If the stationary random process is ergodic, $\left\langle y^2(t) \right\rangle = \overline{y_i^2}$; i = 1, 2, 3, ...

FIGURE 1-2. COMPUTATION OF MEAN SQUARE VALUES

$\langle\ \rangle$ Indicates ensemble average

— Indicates time average



$$\overline{y_1(t)y_1(t+\tau)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T y_1(t)y_1(t+\tau)\,dt$$

= time correlation for record $y_1(t)$

$$\overline{y_k(t)y_k(t+\tau)} = \lim_{T\to\infty} \frac{1}{T}\int_0^T y_k(t)y_k(t+\tau)\,dt$$

= time correlation for record $y_k(t)$

$$\langle y(t_1)y(t_2)\rangle = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} y_i(t_1)y_i(t_2)$$

= ensemble correlation for times $t = t_1$ and t_2

If the random process is stationary, $\langle y(t_1)y(t_2)\rangle = \langle y(t)\,y(t+\tau)\rangle$

If the stationary random process is ergodic,

$$\langle y(t)y(t+\tau)\rangle = \overline{y(t)y(t+\tau)} = R_y(\tau)$$

FIGURE 1-3. COMPUTATION OF AUTOCORRELATION VALUES

completely described only by taking averages over the entire ensemble at every instant of time. If the vibration response is representative of a stationary random process, the properties of the vibration can be described by taking averages over the entire ensemble at any one instant of time. If a vibration response is representative of a stationary and ergodic random process, the properties of the vibration can be described by taking time averages over one record from the ensemble.

For the actual flight vehicle vibration problem, a single record may represent the vibration response at some point on the structure of a given vehicle. The collection of records needed to form an ensemble would then represent the vibration responses at that point occurring during flights of all vehicles of that type. However, data from a large number of vehicles of the same type is rarely available. An ensemble may be contrived by collecting records of the vibration response for repeated flights of the same vehicle, where the time origin of each record is considered to be the start of each flight. Even data of this sort is often difficult to acquire. Usually vibration data from only a few flights or perhaps just one flight is available. As a result, the vast majority of vibration data reduction is performed by time averaging single sample records.

1.2 BASIC DESCRIPTIONS OF VIBRATION DATA

For any time invariant vibration response, whether it be random, periodic, or a combination of both, the simplest description of the vibration amplitude is given by the mean square value. For a vibration record $y(t)$ of length T, the mean square value $\overline{y^2}$ is given by

$$\overline{y^2} = \frac{1}{T}\int_0^T y^2(t)\,dt \tag{1.4}$$

$T = T_p$ for periodic vibration

$T \longrightarrow \infty$ for random vibration

The positive square root of the mean square value gives the rms (root mean square) value of the vibration. That is,

$$y_{rms} = \sqrt{\overline{y^2}} \tag{1.5}$$

The mean square value is a measure of both the static and dynamic portions of the vibration amplitude. The static portion of the vibration (the D.C. component) is defined by the mean value $\bar{y}$ as follows.

$$\bar{y} = \frac{1}{T}\int_0^T y(t)\,dt \tag{1.6}$$

$T = T_p$ for periodic vibration

$T \longrightarrow \infty$ for random vibration

The dynamic portion of the vibration is defined by the mean square value about the mean (variance) σ_y^2 as follows.

$$\sigma_y^2 = \frac{1}{T}\int_0^T \left[y(t) - \bar{y}\right]^2 dt \tag{1.7}$$

$T = T_p$ for periodic vibration

$T \longrightarrow \infty$ for random vibration

It is important to note that these three measures of amplitude are related as follows.

$$\overline{y^2} = \sigma_y^2 + (\bar{y})^2 \qquad (1.8)$$

Hence, if the mean value of the vibration is zero ($\bar{y}=0$), which is often the case in actual practice, the mean square value and variance will be equal ($\overline{y^2} = \sigma_y^2$).

Mean square or rms vibration level measurements give only a rudimentary description of the vibration amplitude. For most engineering applications, a more detailed description of the vibration is required. Such detailed descriptions for periodic and random vibration data will now be discussed.

1.2.1 Periodic Vibration Data

As noted in Section 1.1.1, a periodic vibration response can be completely described by a Fourier series which gives the amplitude, frequency, and phase of all harmonic components of the vibration. However, in actual practice, the amplitudes and frequencies for the harmonic components of a vibration response give a sufficient description for most engineering applications. A knowledge of the associated phase angles are not often required. The description of a periodic vibration response in terms of its harmonic amplitudes and frequencies is given by a discrete frequency spectrum.

A typical discrete frequency spectrum is illustrated in Figure 1-4. Note that each harmonic component appears in the frequency spectrum as a line which has no bandwidth. The peak amplitudes of the components (C_0, C_1, C_2, etc.) are equivalent to the coefficients in the Fourier series for the periodic vibration as shown in Eq. (1.3). The mean value of the

$$\overline{y} = C_0$$

$$\overline{y_i^2} = \frac{1}{2} C_i^2$$

$$\overline{y^2} = C_0^2 + \frac{1}{2} \sum_{i=1}^{\infty} C_i^2$$

$$\sigma_y^2 = \overline{y^2} - (\overline{y})^2 = \frac{1}{2} \sum_{i=1}^{\infty} C_i^2$$

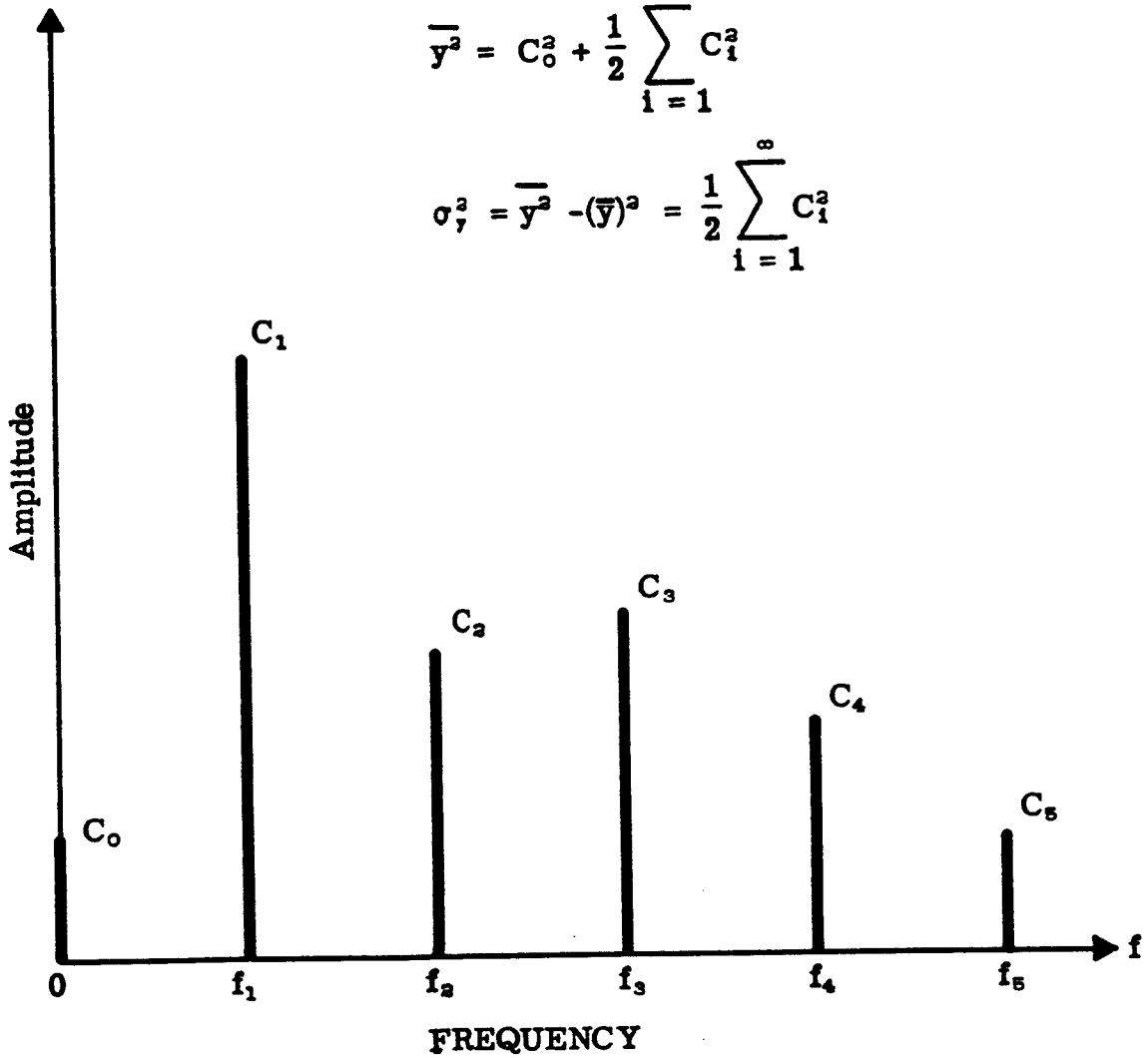


FIGURE 1-4. TYPICAL DISCRETE FREQUENCY SPECTRUM

vibration is defined by the coefficient C_0 at zero frequency. Note that the mean square value of the vibration is equal to the sum of the mean square values for the individual components plus the square of the mean.

1.2.2 Random Vibration Data

If a vibration response is random in nature and assumed to be representative of an ergodic stationary process, a reasonably detailed description of the vibration is obtained from three important properties of random signals. The first is a statistical description of the amplitude characteristics of the vibration, which is called the amplitude probability density function. The second is a statistical description of the time correlation characteristics of the vibration, which is called the autocorrelation function. The third is a statistical description of the frequency composition of the vibration, which is called the power spectral density function. Furthermore, if data from two or more vibration responses is obtained simultaneously, additional information is available from several joint properties. These include joint amplitude probability density functions, cross-correlation functions, cross-power spectral density functions, and coherence functions. These various descriptive properties will now be discussed.

1.2.2.1 Amplitude Probability Density Function

Given a stationary random vibration record $y(t)$ of length T seconds, the first order amplitude probability density function $p(y)$ is as follows.

$$p(y) = \lim_{T \to \infty} \lim_{\Delta y \to 0} \frac{1}{T(\Delta y)} \sum_{i=1}^{\infty} t_i(y, y + \Delta y) \qquad (1.9)$$

The quantity $t_i(y, y+\Delta y)$ is the time spent by the amplitude within the narrow amplitude interval between y and $y+\Delta y$ during the ith entry into the interval.

A typical probability density plot [p(y) versus y] is illustrated in Figure 1-5. The area under the probability density plot between any two amplitudes y_1 and y_2 is equal to the probability of the vibration response having an amplitude within that range at any given time. Obviously, the total area under the plot is equal to unity since the probability of the vibration having any amplitude must be one. In other words, it is certain that the vibration response will have some amplitude between ± infinity. Note that the mean value, mean square value, and variance for the vibration are related to the probability density function as follows.

$$\bar{y} = \int_{-\infty}^{\infty} y\, p(y)\, dy \qquad (1.10a)$$

$$\overline{y^2} = \int_{-\infty}^{\infty} y^2\, p(y)\, dy \qquad (1.10b)$$

$$\sigma_y^2 = \int_{-\infty}^{\infty} (y - \bar{y})^2\, p(y)\, dy = \overline{y^2} - (\bar{y})^2 \qquad (1.10c)$$

1.2.2.2 Autocorrelation Function

Given a stationary random vibration record y(t) of length T seconds, the autocorrelation function $R_y(\tau)$ is as follows.

$$R_y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T y(t)\, y(t+\tau)\, dt \qquad (1.11)$$

The quantity τ is the time difference in seconds, which is often called the lag time. Note that the autocorrelation function is a real valued even function which may be either positive or negative.

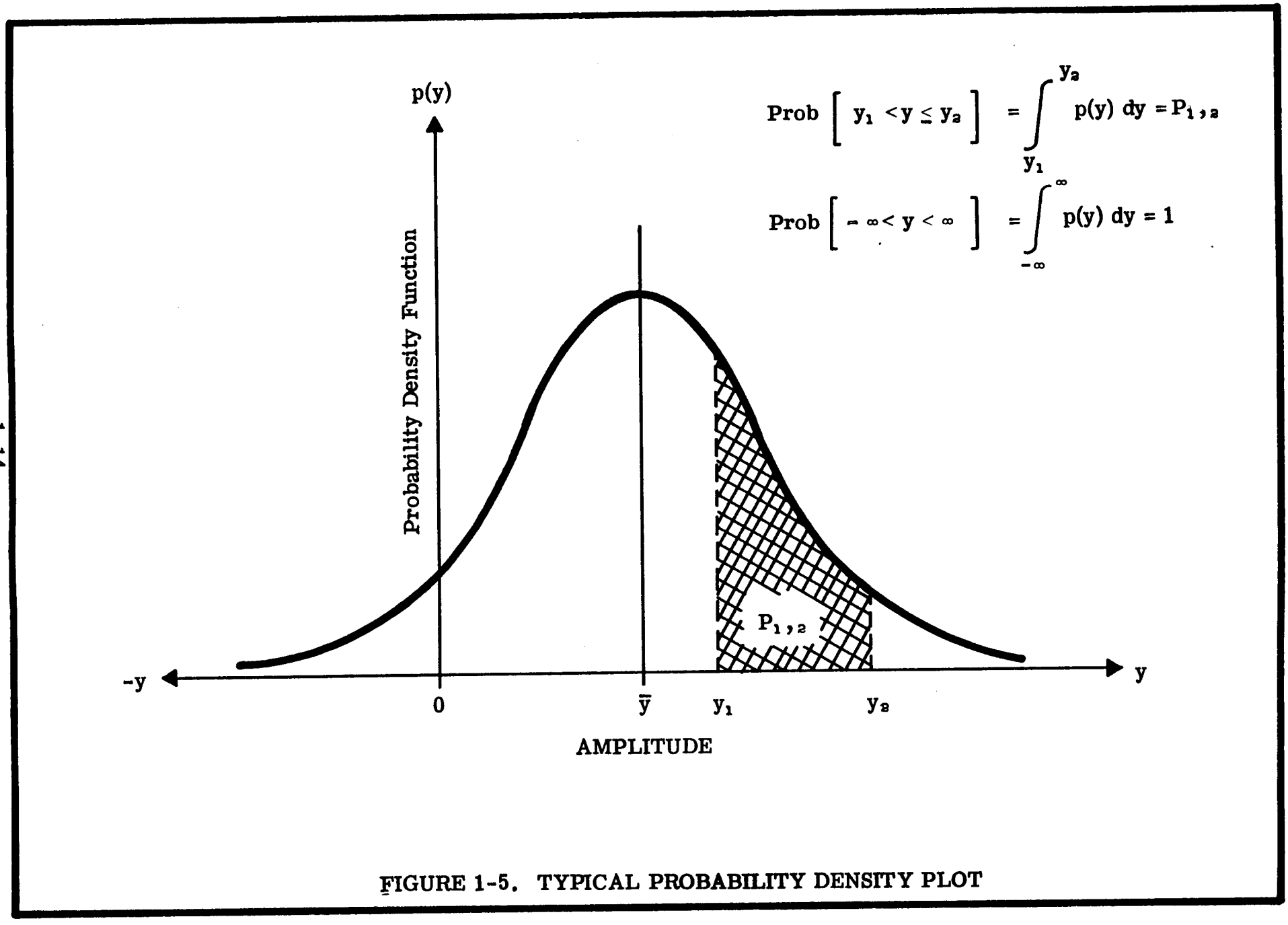


FIGURE 1-5. TYPICAL PROBABILITY DENSITY PLOT

A typical autocorrelation plot $\left[R_y(\tau) \text{ versus } \tau \right]$ is illustrated in Figure 1-6. The autocorrelation function for a vibration response indicates the relative dependence of the vibration amplitude at any instant on the vibration amplitude that had occurred τ seconds before. The maximum value of the autocorrelation function occurs when the lag time is zero. Note that the mean value, mean square value, and variance for the vibration are related to the autocorrelation function as follows.

$$\bar{y} = \sqrt{R_y(\infty)} \qquad (1.12a)$$

$$\overline{y^2} = R_y(0) \qquad (1.12b)$$

$$\sigma_y^2 = R_y(0) - R_y(\infty) \qquad (1.12c)$$

1.2.2.3 Power Spectral Density Function

Given a stationary random vibration record $y(t)$ of length T seconds, the power spectral density function $G_y(f)$ is as follows.

$$G_y(f) = \lim_{T \to \infty} \lim_{\Delta f \to 0} \frac{1}{T(\Delta f)} \int_0^T y^2_{\Delta f}(f,t)\, dt \qquad (1.13)$$

The quantity $y^2_{\Delta f}(f,t)$ is the square of the amplitudes within the narrow frequency interval between f and $f + \Delta f$. Note that the power spectral density function is a real valued function that is always positive.

A typical power spectrum $\left[G_y(f) \text{ versus } f \right]$ is illustrated in Figure 1-7. The area under the power spectrum plot between any two frequencies f_1 and f_2 is equal to the mean square value of the vibration response within that frequency range. The total area under the plot is equal to the total mean

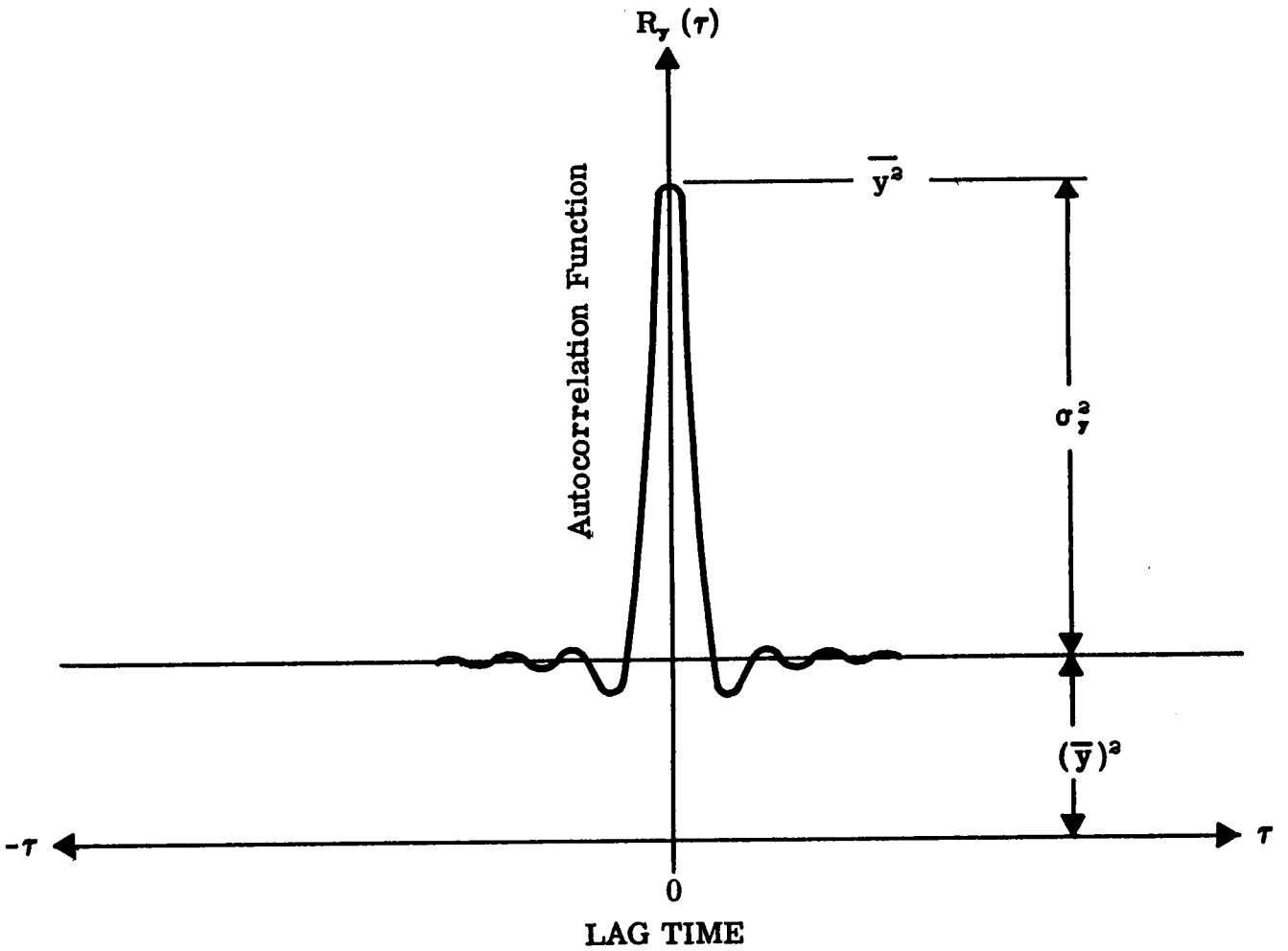


FIGURE 1-6. TYPICAL AUTOCORRELATION PLOT

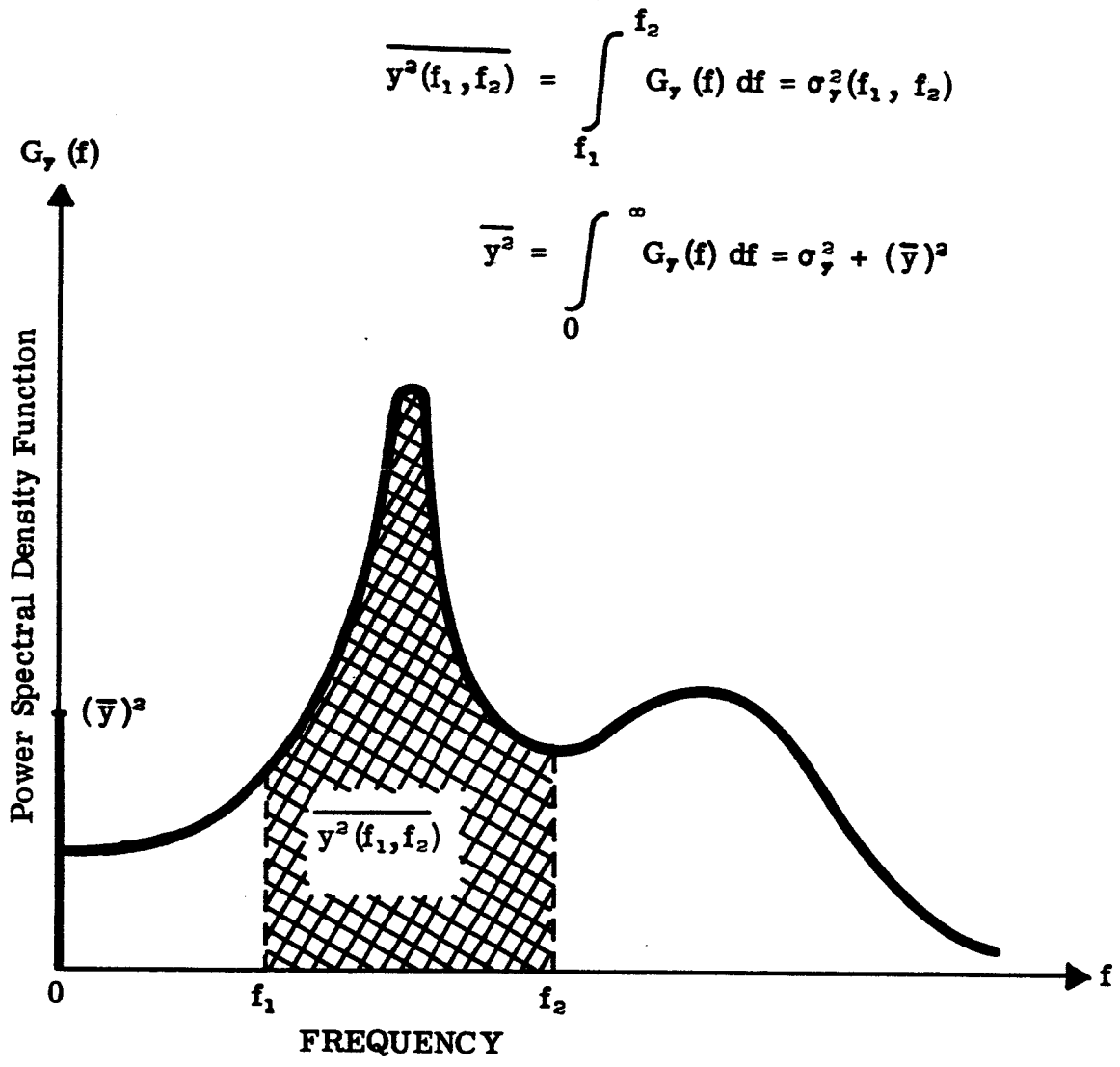


FIGURE 1-7. TYPICAL POWER SPECTRUM

square value of the vibration response. If the vibration response has a non-zero mean value (a D.C. component), this will appear in the power spectrum as a delta function at zero frequency. The area under the delta function is equal to the square of the mean value. Note that the existence of a delta function at a frequency other than zero would represent a sine wave at that frequency.

It is important to mention that the power spectral density function for a stationary random signal is the Fourier transform of the autocorrelation function. Hence, a power spectrum contains the same basic information as an autocorrelation plot. Furthermore, the power spectrum presents the information in a frequency format which is more convenient for most engineering applications. However, there are special situations where an autocorrelation plot is more useful than a power spectrum. An example is the problem of detecting periodic components in an otherwise random vibration response. These matters are discussed in greater depth in Section 2.

1.2.2.4 Joint Amplitude Probability Density Function

Given two stationary random vibration records, x(t) and y(t), each of length T seconds, the joint amplitude probability density function p(x, y) is as follows.

$$p(x, y) = \lim_{T \to \infty} \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{1}{T(\Delta x)(\Delta y)} \sum_{i=1}^{\infty} t_i(x, x+\Delta x; y, y+\Delta y) \qquad (1.14)$$

The quantity $t_i(x, x+\Delta x; y, y+\Delta y)$ is the time spent by the amplitudes x(t) and y(t) when they are simultaneously within the narrow amplitude intervals

between x and $x + \Delta x$, and y and $y + \Delta y$, respectively, during the ith simultaneous entry into the intervals.

A typical joint probability density plot $\left[p(x, y) \text{ versus } x \text{ and } y \right]$ is illustrated in Figure 1-8. Note that the plot has three dimensions. The volume under the joint probability density plot bounded by the amplitudes x_1, x_2, y_1, and y_2 is equal to the probability that $x(t)$ and $y(t)$ will simultaneously have amplitudes within those ranges at any given time. Obviously, the total volume under the plot is equal to unity since the probability of the two vibration responses simultaneously having any amplitudes must be one.

1.2.2.5 Cross-Correlation Function

Given two stationary random vibration records, $x(t)$ and $y(t)$, each of length T seconds, the cross-correlation function $R_{xy}(\tau)$ is as follows.

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, y(t + \tau)\, dt \qquad (1.15)$$

The quantity τ is the time difference in seconds, which is often called the lag time. Note that the cross-correlation function is real valued but not even, and may be either positive or negative.

A typical cross-correlation plot $\left[R_{xy}(\tau) \text{ versus } \tau \right]$ is illustrated in Figure 1-9. The value of the cross-correlation function for two vibration responses indicates the relative dependence of the amplitude of one vibration response at any instant of time on the amplitude of the other vibration response that had occurred τ seconds before. In actual practice, cross-correlation functions have wide applications to the evaluation of linear

$$\text{Prob}\left[x_1 < x \leq x_2;\ y_1 < y \leq y_2\right] = \int_{y_1}^{y_2}\int_{x_1}^{x_2} p(x,y)\,dx\,dy = P_{1,2}$$

$$\text{Prob}\left[-\infty < x < \infty;\ -\infty < y < \infty\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 1$$

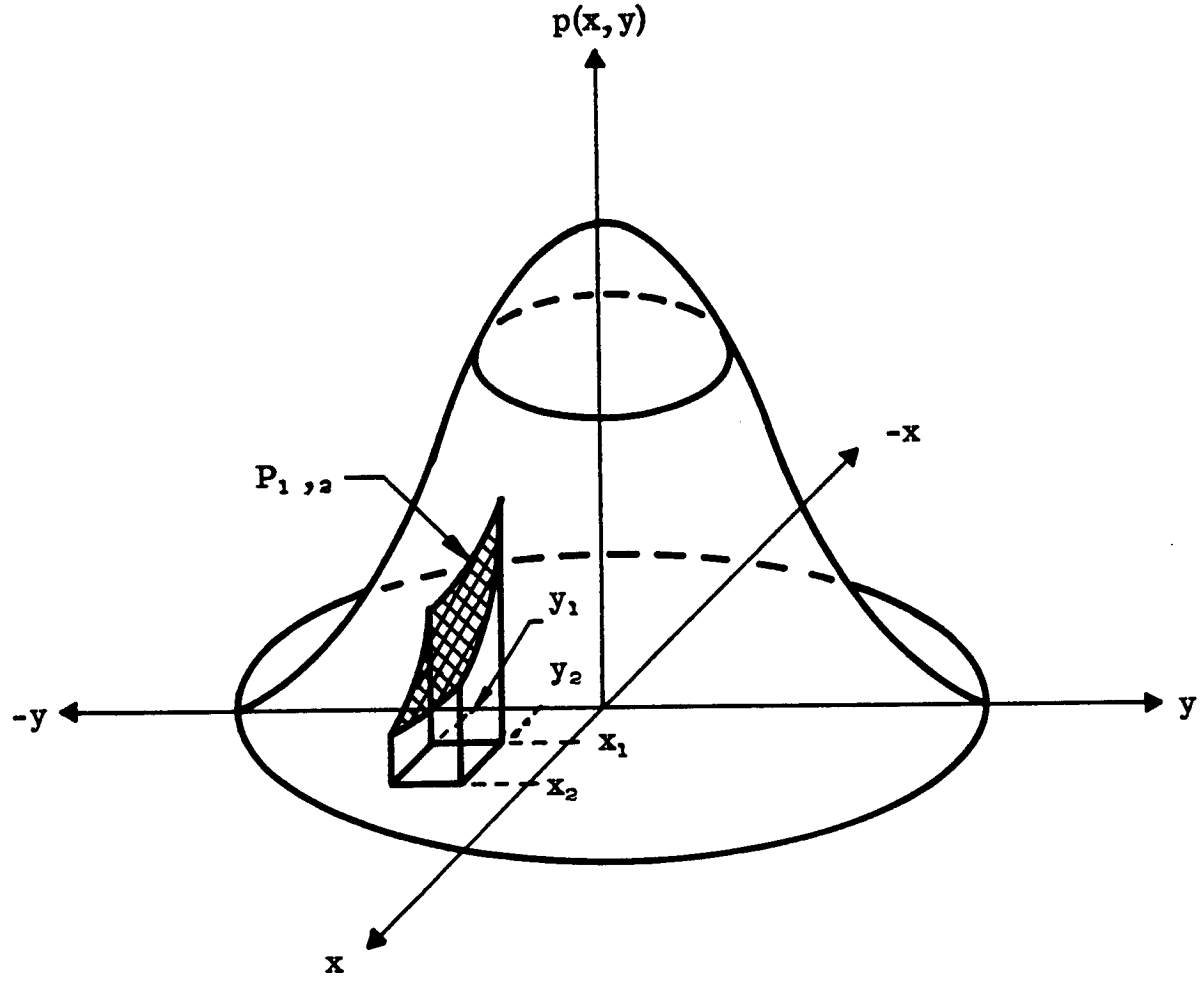


FIGURE 1-8. TYPICAL JOINT PROBABILITY DENSITY PLOT

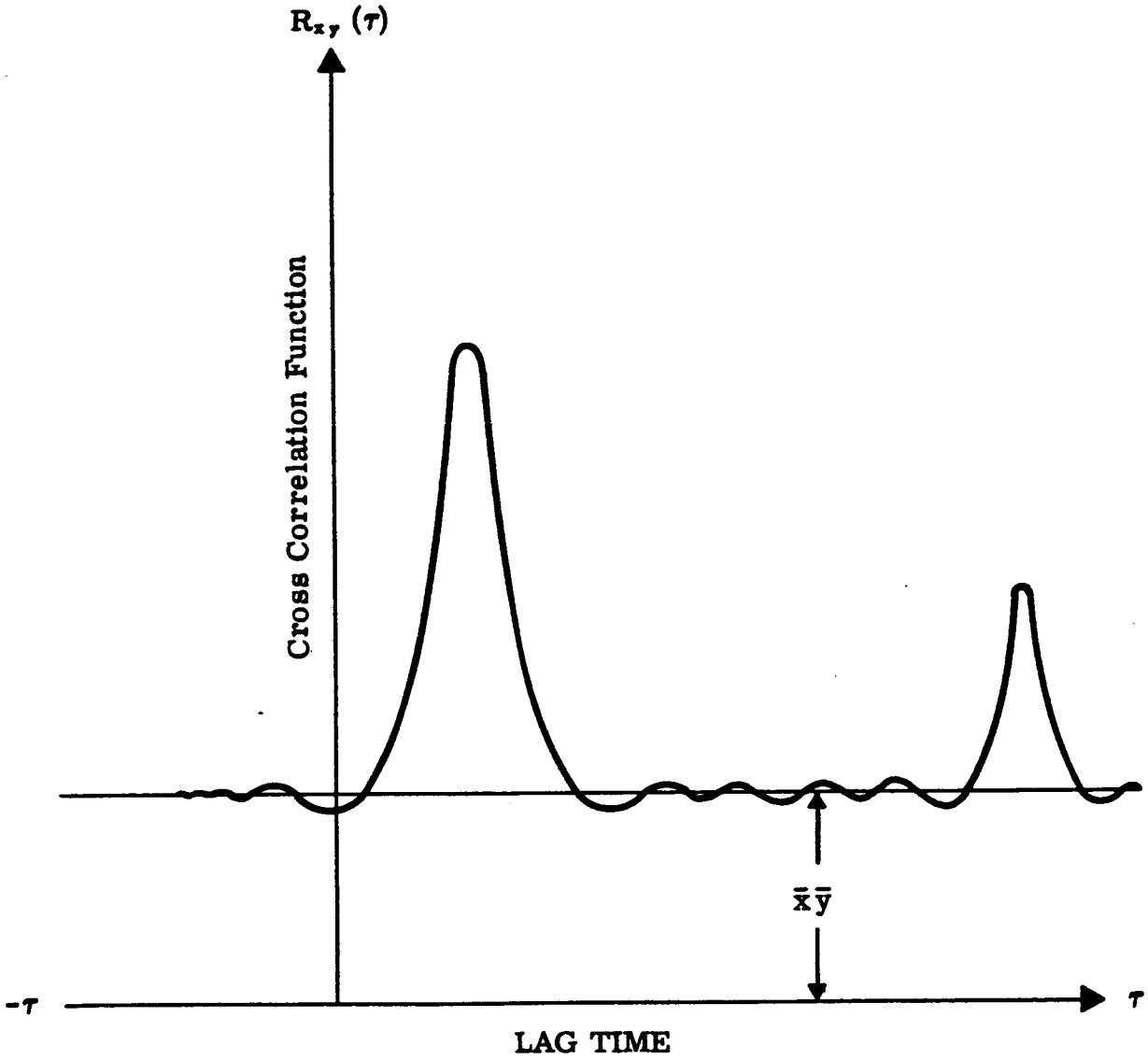


FIGURE 1-9. TYPICAL CROSS CORRELATION PLOT

structural transfer characteristics. Furthermore, cross-correlation functions furnish a powerful tool for localizing vibration sources by determining time delays and structural transmission paths. These matters are discussed in more detail in Section 2.

1.2.2.6 Cross-Power Spectral Density Function

Given two stationary random vibration records, x(t) and y(t), each of length T seconds, the cross-power spectral density function $G_{xy}(f)$ is as follows.

$$G_{xy}(f) = C_{xy}(f) - jQ_{xy}(f) \qquad (1.16)$$

In words, the cross-power spectral density function is a complex-valued function with a real part $C_{xy}(f)$ called the cospectrum, and imaginary part $Q_{xy}(f)$ called the quad-spectrum. The cospectrum and quad-spectrum are given by

$$C_{xy}(f) = \lim_{T\to\infty} \lim_{\Delta f\to 0} \frac{1}{T(\Delta f)} \int_0^T x_{\Delta f}(f, t)\, y_{\Delta f}(f, t)\, dt \qquad (1.16a)$$

$$Q_{xy}(f) = \lim_{T\to\infty} \lim_{\Delta f\to 0} \frac{1}{T(\Delta f)} \int_0^T \tilde{x}_{\Delta f}(f, t)\, y_{\Delta f}(f, t)\, dt \qquad (1.16b)$$

The quantities $x_{\Delta f}(f, t)$ and $y_{\Delta f}(f, t)$ are the amplitudes within the narrow frequency interval between f and $f+\Delta f$. The symbol ($\sim$) in Eq. (1.16b) means that x(t) is 90 degrees out of phase with y(t). Note that both the cospectrum and quad-spectrum may have either positive or negative values.

It is usually more desirable to express the cross-power spectral density function in complex polar coordinates with a magnitude and phase angle, rather

than a "standard" complex number as in Eq. (1.16). In polar notation, the cross-power spectral density function $G_{xy}(f)$ is given by

$$G_{xy}(f) = |G_{xy}(f)| e^{j\theta_{xy}(f)} \tag{1.17}$$

where the magnitude term $|G_{xy}(f)|$ and the associated phase angle $\theta_{xy}(f)$ are as follows.

$$|G_{xy}(f)| = \sqrt{C^2_{xy}(f) + Q^2_{xy}(f)} \tag{1.17a}$$

$$\theta_{xy}(f) = \text{arc tan } \frac{-Q_{xy}(f)}{C_{xy}(f)} \tag{1.17b}$$

A typical cross-power spectrum $\left[G_{xy}(f) \text{ versus } f \right]$ is illustrated in Figure 1-10. Cross-power spectral density functions have wide applications to the measurement and evaluation of structural transfer characteristics. The details of such applications are beyond the scope of this section. However, several important associations involving the cross-power spectrum for structural input-output relationships will be noted.

Consider a structure (or any other physical system) with a <u>linear</u> frequency response function of $H(f)$. Note that $H(f)$ is a complex-valued function. Let the input excitation be $x(t)$ and the output response be $y(t)$. The following formulas apply.

$$G_{xy}(f) = H(f)\, G_x(f) \tag{1.18a}$$

$$G_{yx}(f) = H^*(f)\, G_x(f) \tag{1.18b}$$

$$H(f) = \frac{G_{xy}(f)}{G_x(f)} = \frac{G_y(f)}{G_{yx}(f)} \tag{1.18c}$$

Here, $H^*(f)$ is the complex conjugate of $H(f)$, $G_x(f)$ is the power spectrum of the excitation x(t), and $G_y(f)$ is the power spectrum of the response y(t). It is important to note that $G_{xy}(f) \neq G_{yx}(f)$.

It was mentioned in 1.2.2.3 of this section that the power spectral density function and the autocorrelation function for a stationary vibration response are Fourier transform pairs. It should now be noted that the cross-power spectral density function and the cross-correlation function for two stationary vibration responses are also Fourier transform pairs. Hence, both functions contain the same basic information. In general, the time format of the cross-correlation function is more convenient for investigations of structural transmission paths and time delays while the frequency format of the cross-power spectrum is more convenient for evaluating structural transfer characteristics.

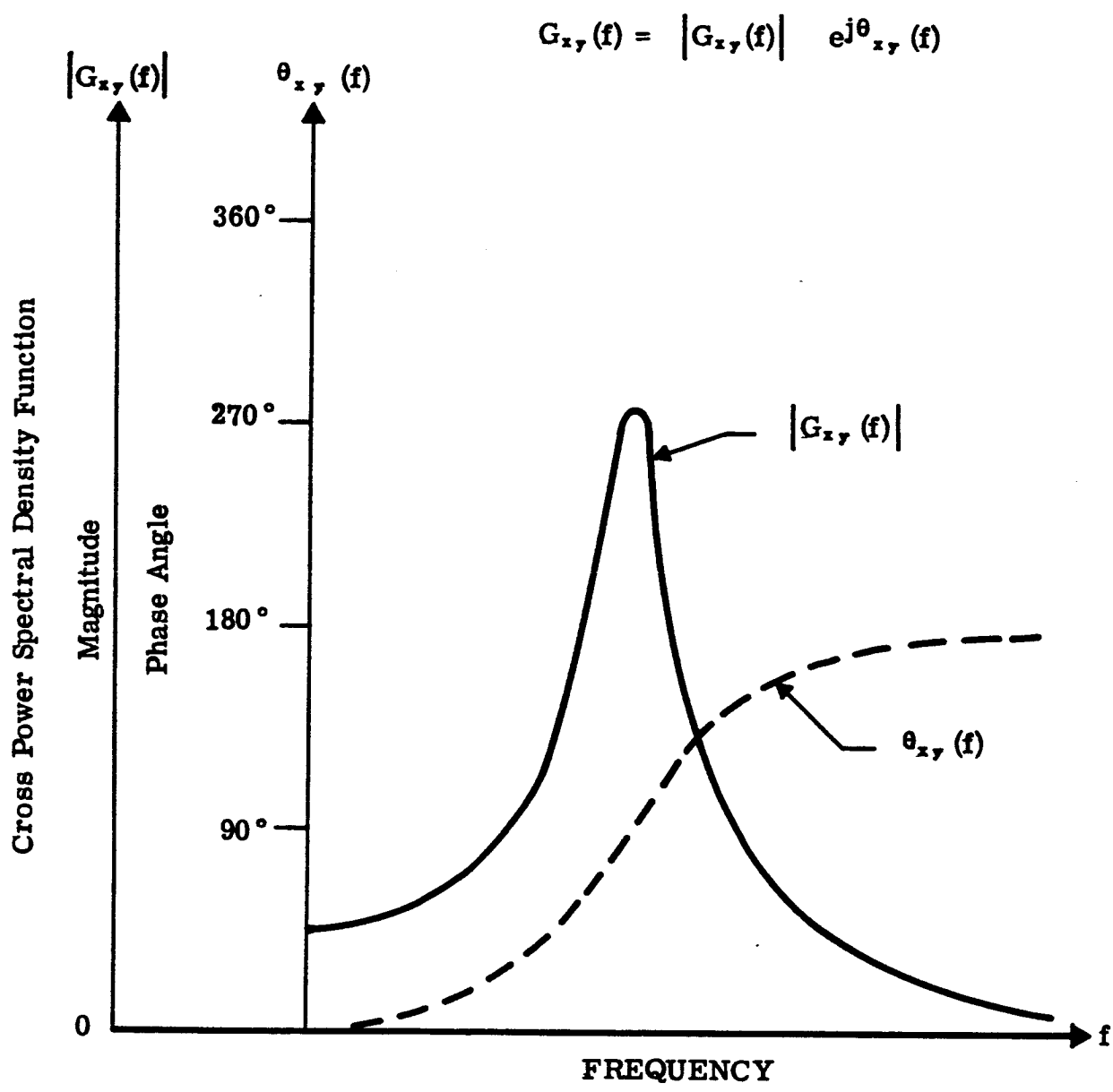


FIGURE 1-10. TYPICAL CROSS POWER SPECTRUM

1.2.2.7 Coherence Function

Given two stationary random vibration records, x(t) and y(t), the coherence function $\gamma^2_{xy}(f)$ is as follows.

$$\gamma^2_{xy}(f) = \frac{G_{xy}(f)\, G_{yx}(f)}{G_x(f)\, G_y(f)} = \frac{|G_{xy}(f)|^2}{G_x(f)\, G_y(f)} \tag{1.19}$$

Here, $G_x(f)$ and $G_y(f)$ are the power spectra of x(t) and y(t), respectively, and $G_{xy}(f)$ and $G_{yx}(f)$ are the cross-power spectra between x(t) and y(t).

If the two vibration responses are completely uncorrelated (incoherent), the coherence function will equal zero. If the two vibration responses are correlated in a linear manner, the coherence function will have a value between zero and one depending upon the degree of correlation. A coherence function of unity means that the vibration measured at one point is the result solely of the vibration at the second point. These matters are discussed in greater depth in Section 2.

1.3 GENERAL TECHNIQUES FOR PERIODIC DATA REDUCTION

The basic analog techniques which are useful for periodic vibration data reduction and analysis are now reviewed in terms of general functions. The basic digital techniques employed for vibration data reduction and analysis are presented in Section 3.6.

As noted in Section 1.2.1, a periodic vibration response can be completely described (except for phase relationships) by a discrete frequency spectrum which gives the amplitude and frequency of all harmonic components. Given a sample vibration response record in the form of an analog voltage signal, a discrete frequency spectrum for the sampled data may be obtained by using an electronic wave analyzer, or as it is often called, a spectrum analyzer.

There are two basic types of spectrum analyzers. The first type employs a collection of contiguous frequency bandpass filters. The filters may be either constant bandwidth filters or constant percentage filters whose bandwidths are proportional to their center frequencies. When a periodic signal is applied to the bank of filters, each passes those frequencies lying within its pass band and excludes all others. The output amplitudes from the filters are then detected and recorded simultaneously as a function of time. The instantaneous output from the filters may also be recorded directly, if so desired. Hence, the spectrum of the applied signal is broken up into as many intervals as there are filters in the bank. This

multiple filter type analyzer is sometimes called a real time spectrum analyzer because its operation is substantially instantaneous. This feature constitutes its primary advantage. A secondary advantage is that phase information can be retained if proper calibration procedures are employed. The primary disadvantage of a multiple filter type analyzer is cost. If high resolution is to be obtained, a large number of expensive filters and amplitude detectors must be incorporated in the analyzer. A functional block diagram for a multiple filter type spectrum analyzer is shown in Figure 1-11.

The second type of spectrum analyzer employs a single narrow frequency bandpass filter. The signal is moved in frequency past the fixed narrow bandpass filter by application of the heterodyne principle. The output amplitude from the filter is detected and recorded as a function of frequency, giving the spectrum for the applied signal. The primary advantage of the single filter type spectrum analyzer is high resolution. Since only a single fixed filter is used, its characteristics can be optimized without adding appreciably to the cost of the analyzer. The primary disadvantage of this single filter type analyzer is that the time required to perform an analysis is relatively long since the entire frequency range of the signal is investigated with only one narrow bandpass filter. It should be noted that single filter spectrum analyzers are usually equipped with several filter selections having different bandwidths to permit flexibility in choosing the resolution desired for a given analysis. A functional block diagram for a single filter type spectrum analyzer is shown in Figure 1-12.

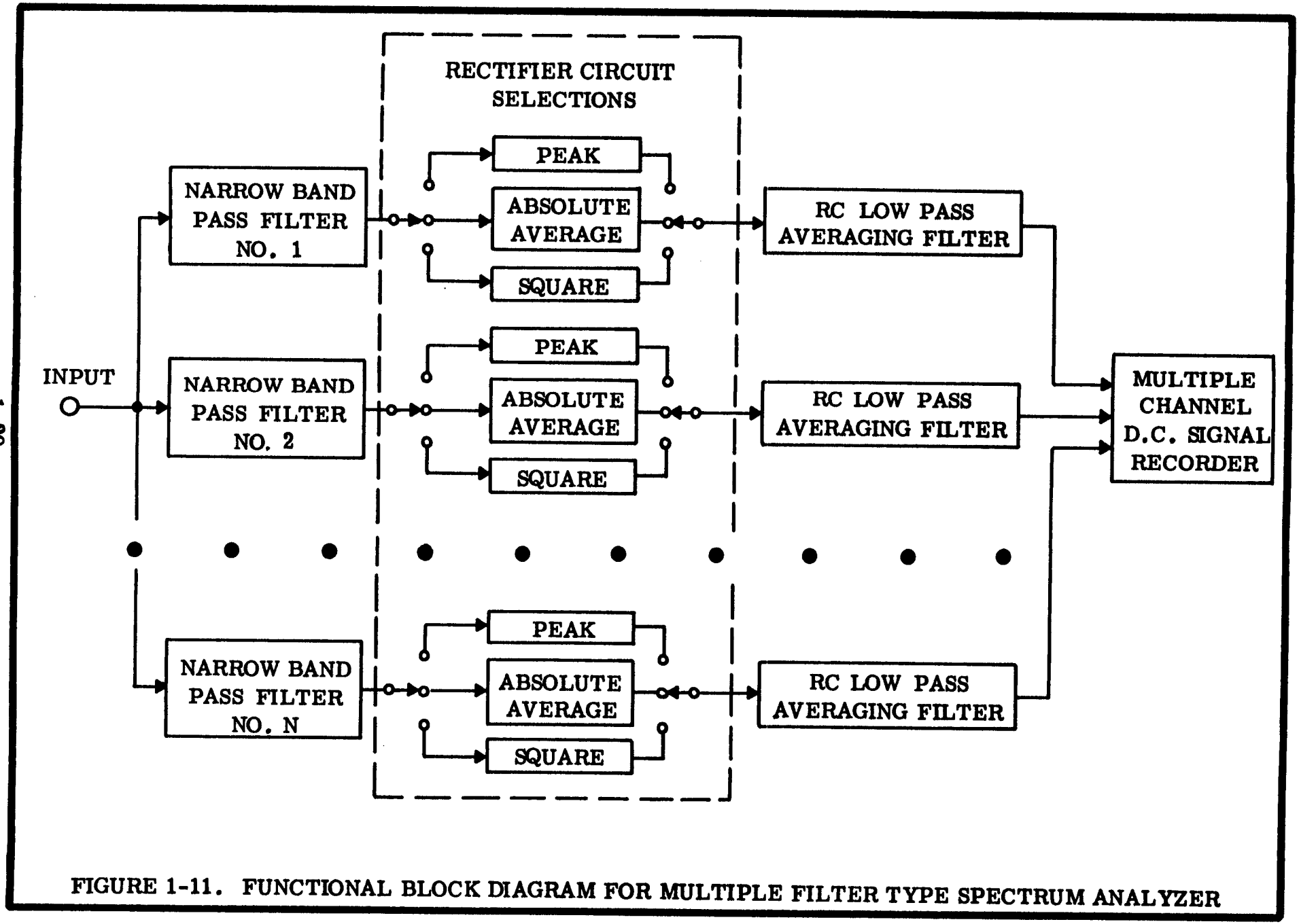


FIGURE 1-11. FUNCTIONAL BLOCK DIAGRAM FOR MULTIPLE FILTER TYPE SPECTRUM ANALYZER

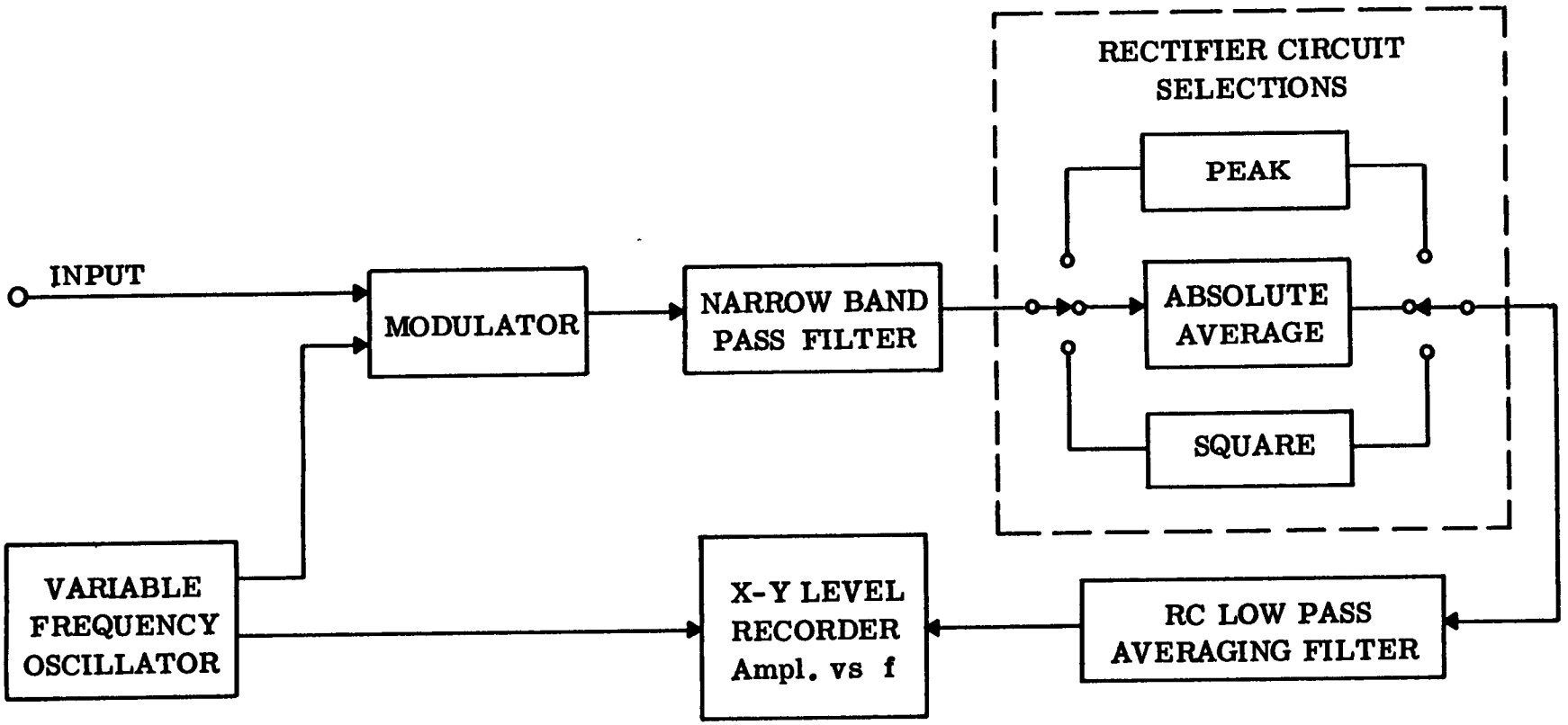


FIGURE 1-12. FUNCTIONAL BLOCK DIAGRAM FOR SINGLE FILTER TYPE SPECTRUM ANALYZER

For either type of spectrum analyzer, the output amplitude detector consists of circuits which compute one or more of three different amplitude functions; namely, the peak amplitude, the rectified average amplitude, and/or the mean square amplitude. If the resolution for a given periodic signal analysis is sufficiently sharp to identify each individual frequency component of the signal, the amplitude detection circuit employed is of no direct concern since the outputs from the filter(s) will always be sine waves. The peak, average, and mean square amplitudes for sine waves have fixed relationships to one another, as follows.

$$\begin{aligned} &\text{instantaneous amplitude} && = C \sin \omega t \\ &\text{peak amplitude} && = C \\ &\text{rectified average amplitude} && = 0.636\, C \\ &\text{mean square amplitude} && = 0.5\, C^2 \\ &\text{root mean square (rms) amplitude} && = 0.707\, C \end{aligned} \qquad (1.20)$$

Hence, any one of the three detection circuits may be used and read out in terms of any other amplitude function desired by simply calibrating the readout scale in an appropriate manner. For example, the peak value of the filtered signal may be detected and read out as an rms amplitude by noting that the rms amplitude is equal to 0.707 times the peak amplitude. It is important to emphasize that these relationships apply only when the analysis resolution is sufficiently sharp to isolate individual frequency components. The relationships in Eq. (1.20) do not apply to the sum of two or more sine waves.

The practical considerations associated with the analysis of periodic signals are reviewed below. All relationships stated are taken from Reference 2.

(a) Analysis Accuracy

If the various limitations noted in (b) through (d) to follow are observed, the only errors in a spectrum analysis of periodic data are the basic measurement errors inherent in the spectrum analyzer design capabilities and calibration techniques. There are no intrinsic statistical uncertainties or sampling errors associated with the proper reduction and analysis of periodic vibration data.

(b) Resolution

The frequency spectrum for a periodic signal is theoretically a discrete line spectrum where each component is a delta function with no bandwidth. However, a spectrum analyzer will display each component as a peak with an apparent bandwidth, which of course will be the bandwidth of the spectrum analyzer filter. Thus, the exact frequency of the signal components will be more accurately defined as the bandwidth of the analyzer filter is made narrower. The accuracy with which the frequencies of individual components are identified is generally referred to as the resolution of the analysis.

It would appear that the best method of analysis would be to use the narrowest possible bandpass filter. However, for the multiple filter type spectrum analyzer, the required number of filters and the associated

cost are inversely proportional to the bandwidth of the filters. For the single filter type spectrum analyzer, the required analysis time is inversely proportional to the bandwidth of the filter, as is discussed later. It is important to note, however, that the resolution of any given spectrum analysis should always be sufficient to distinguish between adjacent frequency components. In other words, the analyzer filter bandwidth should always be narrower than the frequency interval between the components of the signal being analyzed. Thus, the general criteria for minimum permissible resolution is

$$B < (1/T_p) \tag{1.21}$$

where B is the analyzer filter bandwidth in cps and T_p is the period of the vibration data in seconds.

An illustration of a properly resolved spectrum analysis is presented in Figure 1-13. In this example, $T_p = 1/50$ second, so the maximum permissible bandwidth for acceptable resolution would be $B = 50$ cps. However, the actual bandwidth used was $B = 2$ cps resulting in a very precise resolution.

(c) Sample Record Length

Theoretically, the record length required to perform a spectrum analysis on sampled periodic vibration data is only T_p seconds long (the length of one vibration period). However, for certain practical reasons, it is desirable that the sample record be very much longer than one period.

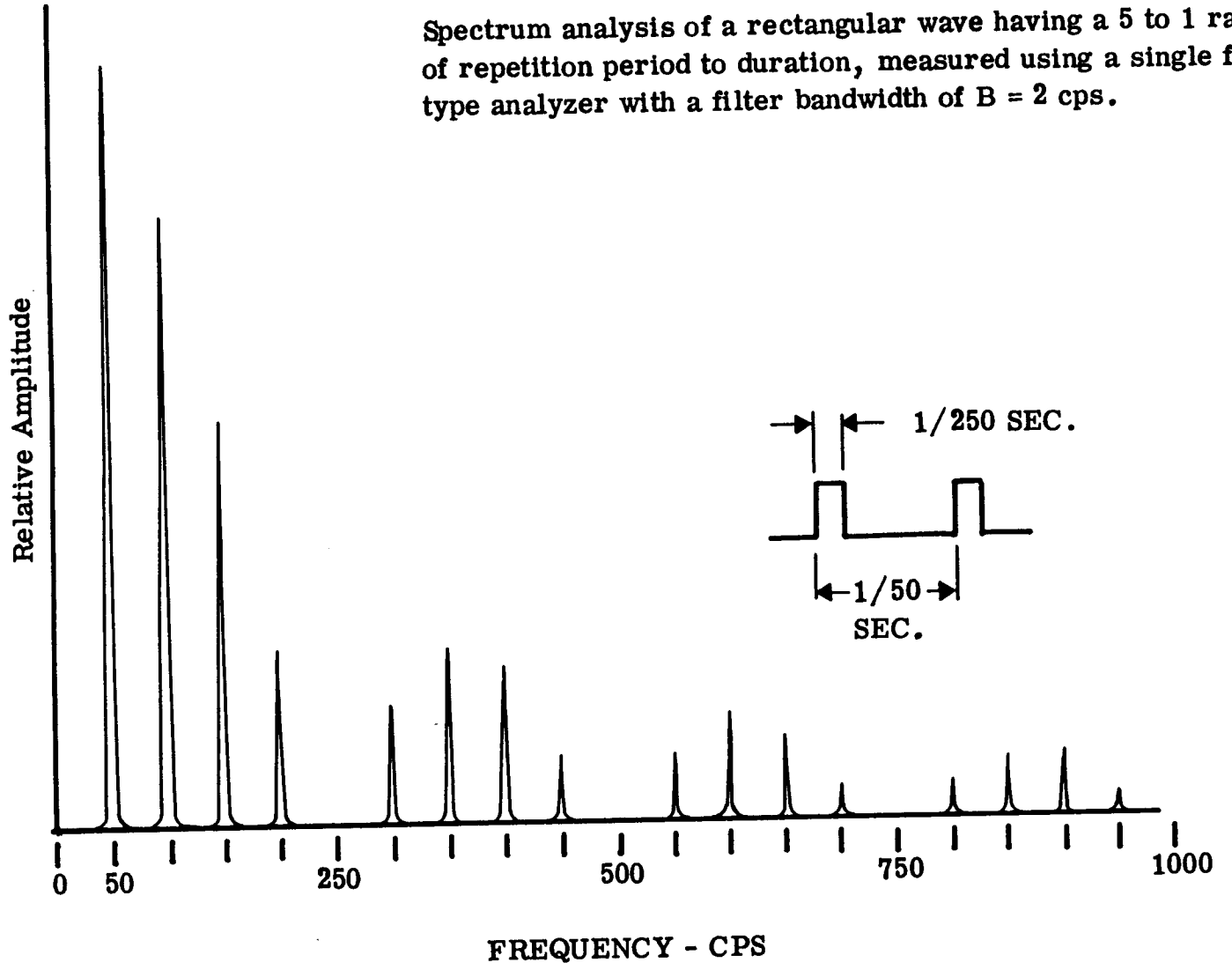


FIGURE 1-13. PROPERLY RESOLVED FREQUENCY SPECTRUM FOR PERIODIC DATA

Multiple filter analyzers are normally employed only when relatively long sample records are involved. However, single filter analyzers are often used to analyze relatively short records.

Analysis of a relatively short sample record with a single filter type spectrum analyzer is usually accomplished by making a continuous loop from the sample record so that the data signal may be continuously applied to the analyzer. The formation of the loop produces, in effect, a fictitious fundamental period for the data. Unless the record length is an exact even multiple of one period T_p, the loop will tend to introduce fictitious frequency components into the analysis. However, these effects become insignificant if the record length is, say, 10 times longer than the period of the vibration data. For example, assume a periodic signal with a fundamental frequency of 25 cps is to be analyzed. The length of the sample record should be $T > 10\,T_p = 0.4$ seconds.

(d) Averaging Time

The peak, average, and/or mean square amplitude detectors incorporated in the spectrum analyzer compute the desired amplitude function by smoothing or time averaging the instantaneous output of an appropriate rectifier circuit. For example, the average amplitude of a signal component is usually measured by a simple A.C. voltmeter circuit where the component is rectified, and the instantaneous rectified amplitude is averaged by smoothing with an equivalent low-pass RC filter. It is clear that the equivalent RC time constant of the averaging filter should be

longer than the period of the vibration data being analyzed. Thus, a general criteria for the minimum time constant of the averaging filter is

$$K \gg T_p \tag{1.22}$$

where K is the equivalent RC time constant in seconds.

For example, assume a periodic signal with a fundamental frequency of 25 cps is to be analyzed. The minimum time constant for the averaging filter is $K \gg T_p = 0.04$ seconds.

(e) Scan Rate and Analysis Time

For the multiple filter type spectrum analyzer, the frequency components of the applied signal are concurrently measured, so the analysis time is substantially instantaneous. However, for the single filter type spectrum analyzer, the frequency components of the applied signal must be individually measured by scanning through the entire frequency range of interest. If the scan is too fast, one of two difficulties may occur.

1. The narrow bandpass filter of the spectrum analyzer will not fully respond to the individual frequency components of the signal.

2. The amplitude detector averaging filter will not fully respond to the individual frequency components of the signal.

The response time for narrow bandpass filters is a function of the exact filter characteristics, but, in general, will be less than $(1/B)$ seconds where B is the bandwidth of the filter in cps. Thus, a general criteria for the maximum analysis scan rate based on the analyzer filter response is

$$\text{scan rate} \ll B^2 \text{ cps/sec} \tag{1.23a}$$

The response time for equivalent RC low-pass averaging filters is such that about 98% of full response is reached in a time equal to four time constants (4K). Thus, a general criteria for the maximum analysis scan rate based on the averaging filter response is

$$\text{scan rate} < \frac{B}{4K} \quad \text{cps/sec} \tag{1.23b}$$

If the total frequency range for the analysis is F cps, the minimum analysis time is

$$\text{analysis time} > \begin{cases} \dfrac{F}{B^2} \text{ seconds} & (1.24a) \\ \dfrac{4KF}{B} \text{ seconds} & (1.24b) \end{cases}$$

For example, assume a periodic signal is to be analyzed with a filter bandwidth of B = 10 cps and an averaging time constant of K = 0.1 seconds, over a frequency range from near zero to 2000 cps (F = 2000 cps). The maximum scan rate is 25 cps/sec, since Eq. (1.23b) produces the smaller value. Hence, the minimum analysis time is 80 seconds, since Eq. (1.24b) produces the larger value.

1.4 GENERAL TECHNIQUES FOR RANDOM DATA REDUCTION

The basic analog techniques which are useful for random vibration data reduction and analysis are now reviewed in terms of general functions. The basic digital techniques employed for vibration data reduction and analysis are presented in Section 3.6.

As noted in Section 1.2.2, a stationary random vibration response can be described in the amplitude domain by a probability density function as given in Eq. (1.9), in the time domain by an autocorrelation function as given in Eq. (1.11), and in the frequency domain by a power spectral density function as given in Eq. (1.13). If two or more vibration response records are available, additional information may be obtained from a joint probability density function as given in Eq. (1.14), a cross-correlation function as given in Eq. (1.15), and a cross-power spectral density function as given in Eq. (1.16).

The true measurement of the above mentioned properties requires the determination of a limit as the record length T approaches infinity. Furthermore, the true measurement of probability and power spectral density functions also requires the determination of a limit as either an amplitude interval Δy or a frequency interval Δf approaches zero. Clearly, the determination of these limits is physically impossible. Thus, no real instrument can actually measure the true properties of a random vibration. However, measurements can be performed which produce

meaningful estimates for the desired properties. These measurement techniques are now discussed.

1.4.1 Amplitude Probability Density Analysis

Given a sample vibration response record in the form of an analog voltage signal $y(t)$ with a finite length of T seconds, the amplitude probability density function for the vibration response may be estimated from Eq. (1.9) as follows.

$$\hat{p}(y) = \frac{\overline{t_W}}{W} \tag{1.25}$$

Here, $\overline{t_W}$ is the average portion of the time spent by the signal $y(t)$ within a narrow amplitude interval having a gate width of W volts and a center amplitude of y volts. The hat (^) over $\hat{p}(y)$ means that the measured quantity is only an estimate of $p(y)$, since the record length T and the gate width W are finite. In words, the amplitude probability density function is estimated by the following operations.

1. Amplitude filtering of the signal by a narrow amplitude gate having a gate width of W volts.
2. Measurement of the total time spent by the signal within the gate.
3. Division of the time spent within the gate by the total sampling time, to obtain the average portion of time spent by the signal within the gate.
4. Division of the average portion of the time spent within the gate by the gate width W.

As the center amplitude of the gate is moved, a plot of the probability density function versus amplitude is obtained.

The above operations are accomplished by an analog amplitude probability density analyzer, which will be called an APD analyzer for simplicity. In general, an APD analyzer measures the time spent by a signal within some narrow amplitude interval by use of a voltage gate (narrow band voltage discriminator) followed by a clock circuit. When the input signal amplitude from the sample record falls within the gate, the clock circuit operates. For all other signal amplitudes, the clock circuit does not operate. The clock circuit output is averaged over the entire time of observation T_a to obtain the average portion of time spent by the signal amplitude within the narrow gate. The required division by the gate width W may be obtained by a proper scale calibration.

There are two basic types of APD analyzers. The first type employs a collection of contiguous voltage gates with equal gate widths. The multiple gate type analyzer measures the probability density within each gate simultaneously to give a plot of probability density versus amplitude. The second type employs a single gate whose center voltage is variable relative to the voltage of the signal. The single gate type analyzer produces a plot of probability density versus amplitude by sweeping (or stepping) the single gate through the entire range of voltage amplitudes of interest. A functional block diagram for a single gate APD analyzer is shown in Figure 1-14.

The practical considerations associated with amplitude probability density analysis are reviewed below. All relationships stated are taken

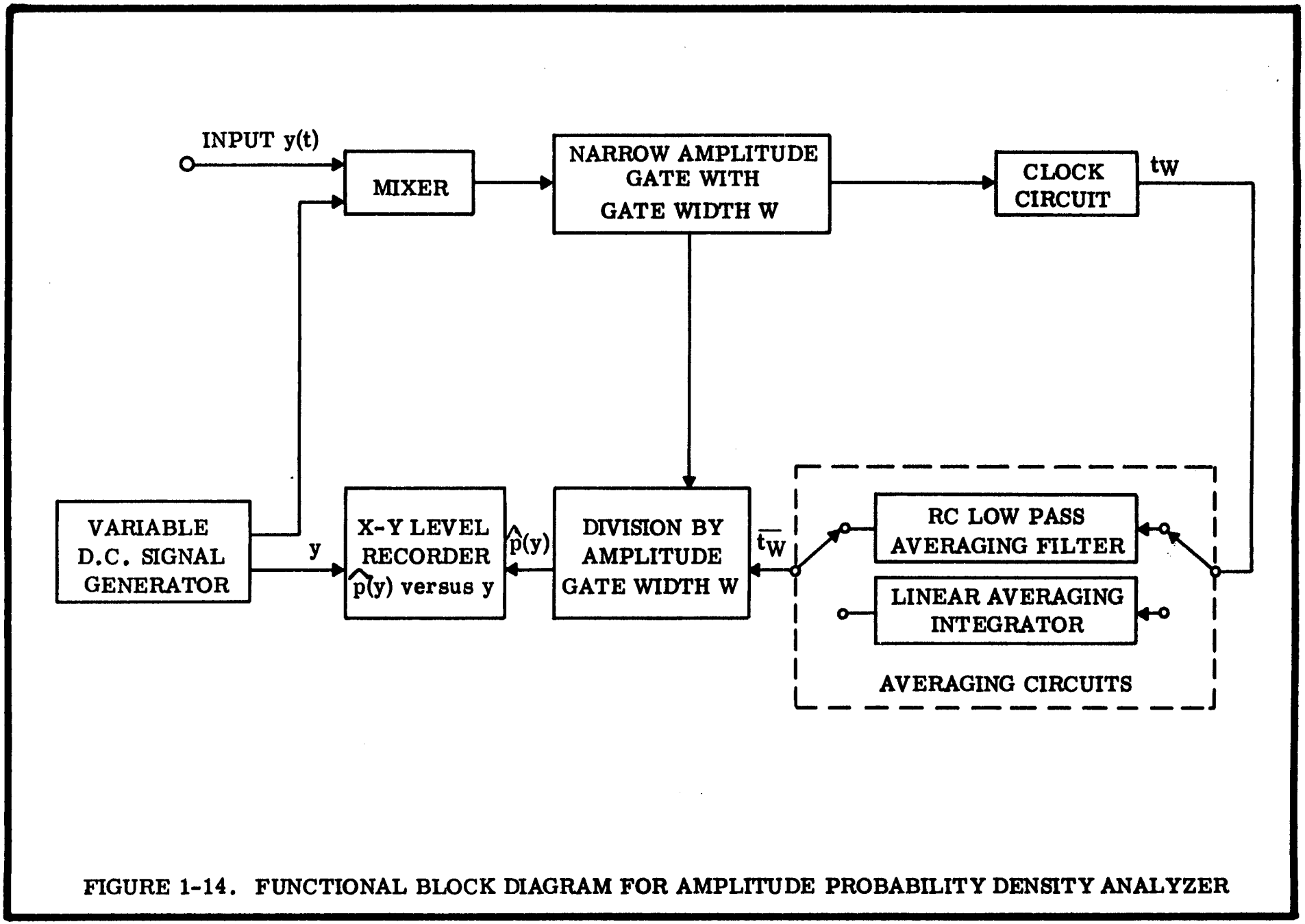


FIGURE 1-14. FUNCTIONAL BLOCK DIAGRAM FOR AMPLITUDE PROBABILITY DENSITY ANALYZER

from Reference 2. Many of the relationships are also studied experimentally in Reference 3.

(a) Analysis Accuracy

The analysis of random vibration data involves basic measurement errors due to the analysis equipment design capabilities and calibration techniques, just as is true for the analysis of periodic vibration data. However, the analysis of random data also involves an additional error which is called the statistical uncertainty or probable sampling error. As was mentioned earlier, the data measured from sample records of finite length constitute only statistical estimates of the true properties of the sampled vibration response. The expected deviation of the estimated properties from the actual properties of the random vibration represent the statistical uncertainties associated with the measurements. This uncertainty may be defined in terms of a normalized standard deviation for the sampling distribution, which is often called the standard error e.

For the specific case of amplitude probability density analysis, the standard error associated with a measured estimate $\hat{p}(y)$ is as follows.

$$e \approx \frac{0.20}{\sqrt{\hat{p}(y)\ WBT}} \qquad (1.26)$$

Here, $\hat{p}(y)$ is the measured probability density, W is the amplitude gate width in relative amplitude units, B is the noise bandwidth of the signal being investigated, and T is the length of the analyzed sample record in seconds.

Several important features of Eq. (1.26) should be noted. First, this expression for the standard error is a simplification of more complicated relationships developed theoretically and empirically in Section 14 of Reference 3. However, Eq. (1.26) is an acceptable approximation for most applications. Second, the standard error e is a function of the actual probability density estimate that is measured. Thus, for any given gate width, bandwidth, and record length, the uncertainty of the estimate varies with the amplitude being analyzed. Third, the bandwidth B is the noise bandwidth of the vibration signal. That is, B is the bandwidth of an ideal rectangular shaped filter which would pass white noise with the same total power as is represented by the vibration signal. The determination of noise bandwidths for random signals is discussed in Section 1.5.

The meaning of the standard error e is as follows. Assume a stationary random vibration response with a true probability density function of $p(y)$ is repeatedly sampled at different times, and an estimate $\hat{p}(y)$ is measured for each sample. For about 68% of the estimates obtained, the difference between the estimate $\hat{p}(y)$ and the true value $p(y)$ will be less than $\pm e\,\hat{p}(y)$. Stated in another way, if an estimate $\hat{p}(y)$ is measured, one may say with about 68% confidence that the true value $p(y)$ is within the range $(1 \pm e)\,\hat{p}(y)$. A plot of the standard error e versus the WBT product for various amplitude probability density estimates is presented in Figure 1-15.

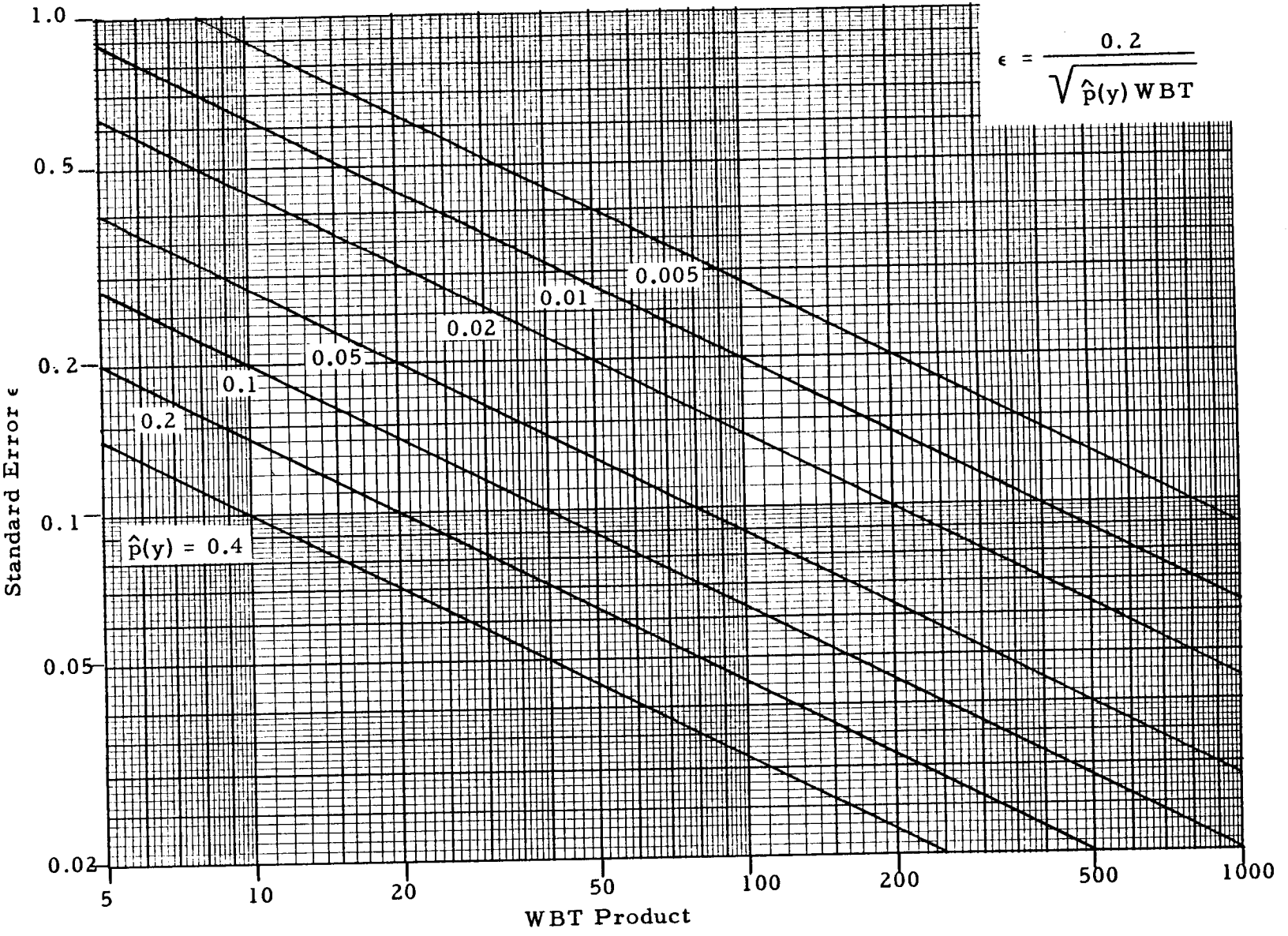


Figure 1-15. Estimation Uncertainty Versus WBT Product

For example, assume an amplitude probability density function is measured from a sample record that is T = 10 seconds long using an APD analyzer with a gate width of W = 0.1 volts. Further assume the noise bandwidth of the signal is B = 100 cps. If a probability density of $\hat{p}$ = 0.16 were measured at the amplitude y = 1.5 volts (y_{rms} = 1 volt), the standard error for the estimate would be e = 0.05. Hence, one could say with 68% confidence that the true probability density at that amplitude is within ± 5% of the measured value, or between 0.152 and 0.168.

(b) Resolution

It is seen in Eq. (1.26) that the statistical uncertainty of probability density estimates is inversely proportional to the width of the amplitude gate. One might then conclude that improved estimation accuracy can easily be obtained by simply increasing the gate width. However, increasing the gate width reduces the resolution of the analysis, i.e., it reduces the ability of the analysis to properly define peaks in the probability density plot. The selection of the analyzer gate width is always a compromise between estimation uncertainty and resolution. A general criteria for proper resolution is a gate width that is less than one-fourth of the rms signal amplitude. That is,

$$W < \frac{1}{4} y_{rms} \tag{1.27}$$

where W is the analyzer gate width in volts and y_{rms} is the root mean square value of the signal being analyzed in volts.

(c) Sample Record Length

As seen from Eq. (1.26), the sample record length T limits the statistical accuracy attainable in an amplitude probability density analysis. The longer the record length, the lower the uncertainty in the resulting probability density estimates. If the statistical uncertainty of a probability density analysis is to be limited to a given desired amount, these matters must be considered before the data is gathered to assure that sample reoords are sufficiently long.

(d) Averaging Time

An APD analyzer computes the portion of time spent by the signal amplitude within the gate by averaging the output of the gate clock circuit. The averaging may be accomplished by true linear integration, called true averaging, or by continuous smoothing with an equivalent low-pass RC filter, called RC averaging. True averaging produces a single probability density estimate after a specific averaging time T_a while RC averaging produces a continuous probability density estimate. If the RC time constant of the averaging filter is K and the record length T is long compared to K, the continuous estimate at any instant of time has an uncertainty equivalent to an estimate obtained by true averaging over a time interval of $T_a = 2K$.

For the case of true averaging, it is clear that the averaging time T_a should be as long as the record length T if the uncertainty in the resulting estimates is to be kept at a minimum. In other words, all of the information available from the sample record should be employed for

the probability density measurement. If T_a is less than T, the uncertainty will be increased since T_a will replace T in Eq. (1.26). If T_a is greater than T, as it could be when the sample record is formed into a continuous loop for analysis, the uncertainty will not be decreased from the value given in Eq. (1.26) since one is simply looking at the same information more than once.

For the case of RC averaging, minimum uncertainty can be achieved only by making the time constant K very long. However, a long averaging time constant reduces the scan rate and greatly increases the total analysis time, as is discussed in (e) to follow. A reasonable compromise is to use an averaging time constant that is at least as long as one record length T. This will produce a continuous measurement which at any instant has a standard error ϵ within 4% of the minimum attainable value given by Eq.(1.26).

Thus, the general criteria for the ideal averaging time for a probability density analysis is as follows.

$$\text{for true averaging,} \quad T_a = T \tag{1.28a}$$

$$\text{for RC averaging,} \quad K \geq T \tag{1.28b}$$

Here, T_a is the true averaging (integration) time in seconds, and K is the time constant of the equivalent RC averaging filter in seconds.

(e) Scan Rate and Analysis Time

For the multiple gate type APD analyzer, the probability density is concurrently measured over all amplitudes of interest, so the analysis time is equal to the record length T. However, for the single gate type APD analyzer, the probability density at all amplitudes of interest must be

measured by scanning through the desired amplitude range. If the scan is too fast, all the information available at a given amplitude will not be viewed by the analyzer gate over the entire record length, and the statistical uncertainty of the resulting estimate will be increased. If RC averaging is used, the scan rate is further limited because time must be allowed for the RC averaging filter to respond to abrupt changes in the probability density function. The limitations imposed upon the scan rate by these considerations are as follows.

$$\text{for true averaging, scan rate} < \frac{W}{T_a} \tag{1.29a}$$

$$\text{for RC averaging, scan rate} < \frac{W}{4K} \tag{1.29b}$$

Here, T_a is the averaging time in seconds, K is the RC time constant in seconds, and W is the gate width in volts. Hence, scan rate has the units of volts per second.

If the total amplitude range for the APD analysis is A volts, the minimum analysis time is as follows.

$$\text{for true averaging, analysis time} > \frac{T_a A}{W} \tag{1.30a}$$

$$\text{for RC averaging, analysis time} > \frac{4KA}{W} \tag{1.30b}$$

For example, assume the amplitude probability density function for a random vibration response is to be estimated from a sample record of length $T = 10$ seconds over an amplitude range from minus 4 volts to

plus 4 volts (A = 8 volts) using an APD analyzer with a gate width of W = 0.1 volts. The rms amplitude of the signal is assumed to be one volt. If true averaging is used, T_a = 10 seconds and the maximum scan rate is 0.01 volts/second. Hence, the minimum analysis time is 800 seconds or 13.3 minutes. If RC averaging is used, $K \geq 10$ seconds, and the maximum scan rate is 0.0025 volts/second. Hence, the minimum analysis time is 3200 seconds, or about 53 minutes.

1.4.2 Autocorrelation Analysis

Given a sample vibration response record in the form of an analog voltage signal y(t) with a finite length of T seconds, the autocorrelation function for the vibration response may be estimated from Eq. (1.11) as follows.

$$\hat{R}_y(\tau) = \overline{y(t)\, y(t+\tau)} \qquad (1.31)$$

Here, $\overline{y(t)\, y(t+\tau)}$ is the average product of the instantaneous signal amplitude at two different times which are τ seconds apart. The hat ($\wedge$) over $\hat{R}_y(\tau)$ means that the measured quantity is only an estimate of $R_y(\tau)$, since the record length T is finite. In words, the autocorrelation function is estimated by the following operations.

1. Delaying the signal by a time displacement equal to τ seconds, called the lag time.
2. Multiplying the amplitude at any instant by the amplitude that had occurred τ seconds before.
3. Averaging the instantaneous amplitude product over the sampling time.

As the lag time is moved, a plot of the autocorrelation function versus lag time is obtained.

The above operations are accomplished by an analog autocorrelation function analyzer, which will be called an ACF analyzer for simplicity. In general, an ACF analyzer displaces the signal in time by use of a magnetic signal recorder with a variable lag time between the record and playback. This can be accomplished, for example, with a magnetic drum recorder where the location of the playback head is variable relative to the location of the record head. The input and output of the lag time generator are then multiplied and averaged. The lag time is variable over a range from zero to the longest sampling times that are anticipated. Since the autocorrelation function is an even function, it is not necessary to make measurements with negative lag times. A functional block diagram for an ACF analyzer is shown in Figure 1-16.

The practical considerations associated with autocorrelation analysis are reviewed below. All relationships stated are taken from Reference 2

(a) Analysis Accuracy

As discussed in Section 1.4.1(a), the analysis of random vibration data involves not only basic measurement errors, but also a statistical uncertainty inherent in the sampling procedures. This uncertainty may be defined in terms of the standard error e for the sampling distribution. For the specific case of autocorrelation analysis, the standard error

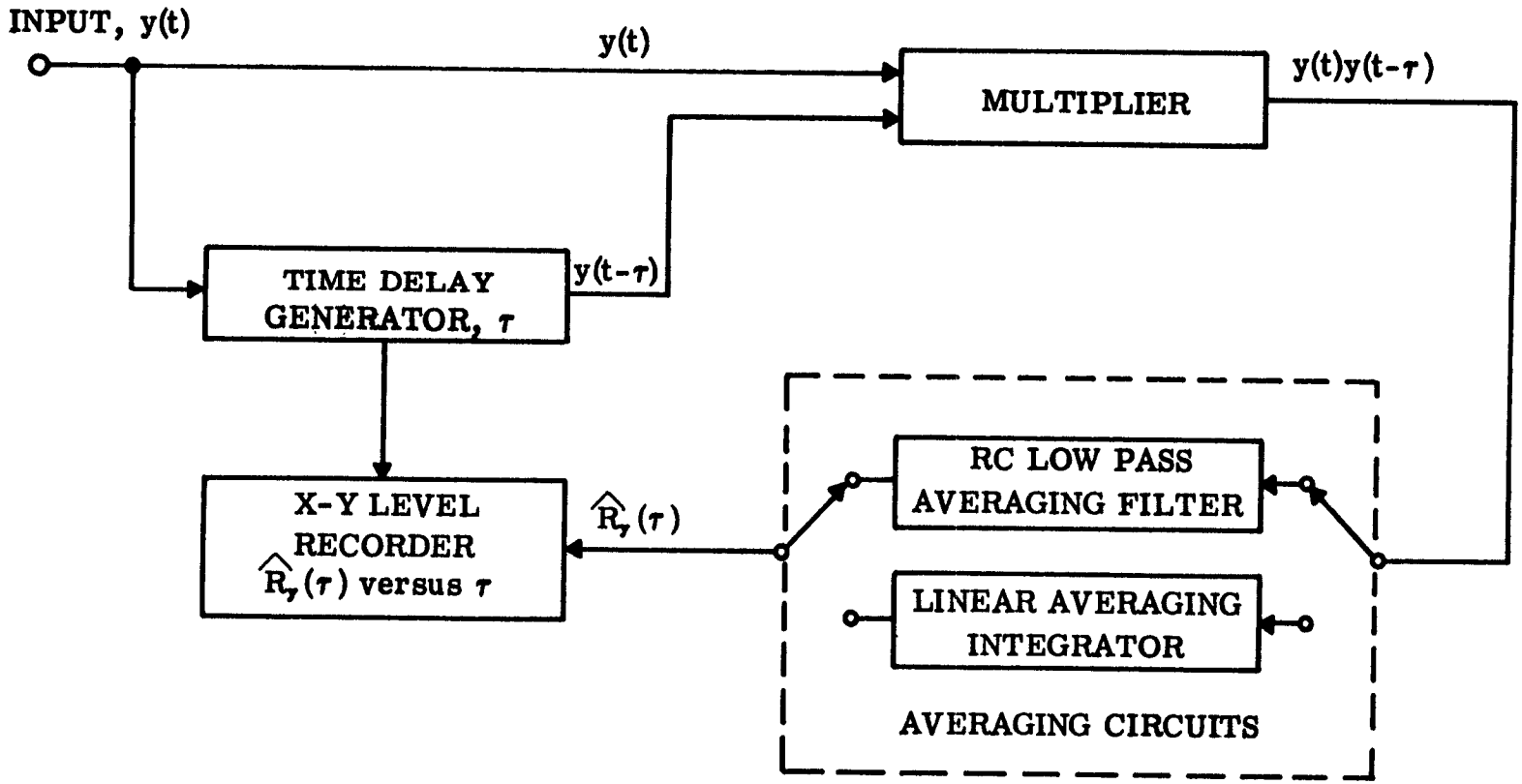


FIGURE 1-16. FUNCTIONAL BLOCK DIAGRAM FOR AUTOCORRELATION ANALYZER

associated with a measured estimate $\hat{R}_y(\tau)$ is as follows.

$$e \approx \frac{1}{\sqrt{BT}} \tag{1.32}$$

Here, B is the noise bandwidth of the signal in cps and T is the length of the analyzed sample record in seconds. It is assumed in deriving Eq. (1.32) that $T \ll |\tau|$ and that $BT \geq 10$.

Two important features of Eq. (1.32) should be noted. First, this expression for the standard error is a conservative approximation which is accurate for lag times near zero. For large lag times, the standard error is not explicitly defined but is somewhat less than the quantity given in Eq. (1.32). Second, the bandwidth B is the noise bandwidth of the vibration response signal. The determination of noise bandwidths is covered in Section 1.5.

The general meaning and interpretation of the standard error e is discussed in Section 1.4.1(a). The specific interpretation for autocorrelation analysis is as follows. Assume a stationary random vibration response with a true autocorrelation function of $R_y(\tau)$ is sampled, and an estimate $\hat{R}_y(\tau)$ is measured from the sample. If e is reasonably small, say less than 0.30, it may be said with about 68% confidence that the true value $R_y(\tau)$ is within the range $(1 \pm e)\hat{R}_y(\tau)$. A plot of the standard error e versus the BT product is presented in Figure 1.17.

For example, assume an autocorrelation function is measured from a sample record that is T = 10 seconds long. Further assume the noise bandwidth of the signal is B = 100 cps. The standard error for the

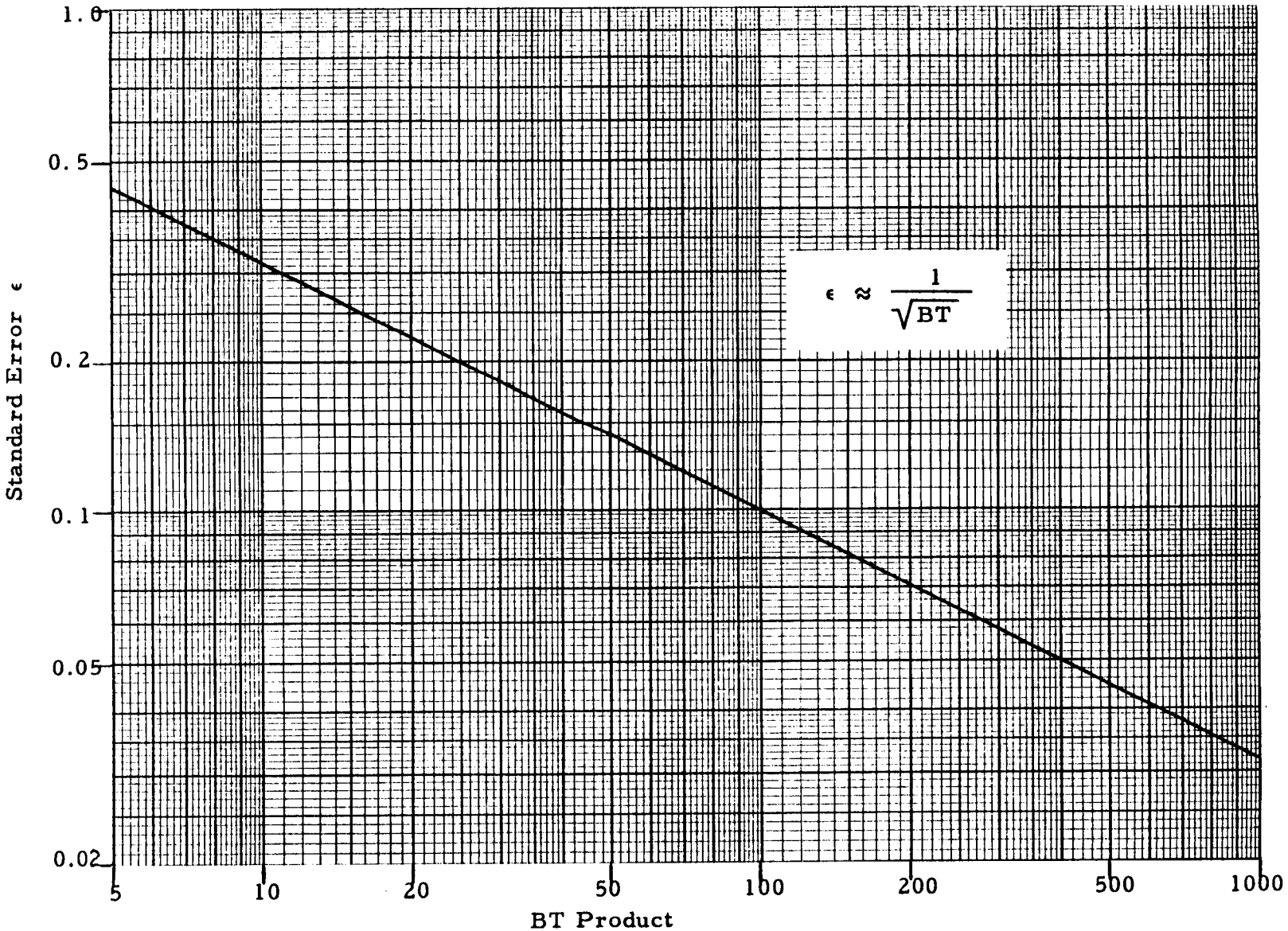


Figure 1-17. Estimation Uncertainty Versus BT Product

resulting estimate is $e \approx 0.032$. Hence, if a measured estimate $\hat{R}_y(\tau)$ at a given lag time were 0.3 volts^2, it could be said with 68% confidence that the true autocorrelation function for that lag time is within $\pm$ 3.2% of the measured value, or between 0.29 and 0.31 volts^2.

(b) Resolution

As seen from Eq. (1.31), the autocorrelation function $\hat{R}(\tau)$ must be estimated at various different lag times τ to obtain a plot of the autocorrelation function versus lag time. The interval between the lag times at which computations are made defines the resolution of the autocorrelation plot. Based upon practical considerations, a general criteria for proper resolution is a lag time interval that is less than one-fourth the reciprocal of the signal bandwidth. That is,

$$h < \frac{1}{4B} \tag{1.33}$$

where h is the interval between lag times in seconds and B is the noise bandwidth of the signal in cps.

The relationship in Eq. (1.33) is directly appropriate for the case when an autocorrelation function is computed at specific lag times, τ_1, $\tau_2 = \tau_1 + h$, $\tau_3 = \tau_1 + 2h$, etc. However, analog instruments often determine an autocorrelation plot by continuous averaging while the time delay generator makes a continuous scan through the lag time range of interest. For this case, the relationship in Eq. (1.33) constitutes the basis for a limit on the lag time scan rate. This limit is discussed in (e) to follow.

(c) Sample Record Length

As seen from Eq. (1.32), the sample record length T determines the statistical accuracy attainable in an autocorrelation analysis. The longer the record length, the lower the uncertainty in the resulting autocorrelation estimates. If the statistical uncertainty of an autocorrelation analysis is to be limited to a given desired amount, these matters must be considered before the data is gathered to assure that sample records are sufficiently long.

(d) Averaging Time

An ACF analyzer computes the mean product of the signal amplitudes at two different times by averaging the output of the multiplier circuit. The averaging may be accomplished by true linear integration, called true averaging, or by continuous smoothing with an equivalent low-pass RC filter, called RC averaging. True averaging produces a single autocorrelation estimate after a specific averaging time T_a while RC averaging produces a continuous probability density estimate.

For the reasons presented in Section 1.4.1(d), the general criteria for the ideal averaging time for an autocorrelation analysis is

$$\text{for true averaging,} \quad T_a = T \tag{1.34a}$$

$$\text{for RC averaging,} \quad K \geqq T \tag{1.34b}$$

Here, T_a is the true integrating time in seconds, and K is the time constant of the equivalent RC averaging filter in seconds.